

THE SELF-DUALITY EQUATIONS ON A RIEMANN SURFACE

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Introduction

In this paper we shall study a special class of solutions of the self-dual Yang–Mills equations. The original self-duality equations which arose in mathematical physics were defined on Euclidean 4-space. The physically relevant solutions were the ones with finite action—the so-called ‘instantons’. The same equations may be dimensionally reduced to Euclidean 3-space by imposing invariance under translation in one direction. These equations also have physical relevance—the solutions which have finite action in three dimensions are the ‘magnetic monopoles’. If we take the reduction process one step further and consider solutions which are invariant under two translations, we obtain a set of equations in the plane. Here, however, there is no clear physical meaning and, indeed, attempts to find finite action solutions have failed. Nevertheless, these are the equations we shall consider.

Despite the lack of interesting solutions in \mathbb{R}^2 , the equations have the important property—*conformal invariance*—which allows them to be defined on manifolds modelled on \mathbb{R}^2 by conformal maps, namely Riemann surfaces. We shall consider here solutions of the self-duality equations defined on a *compact Riemann surface*. There are in fact solutions, as we shall show, and the moduli space of all solutions turns out to be a manifold with an extremely rich geometric structure which will be the focus of our study. It brings together in a harmonious way the subjects of Riemannian geometry, topology, algebraic geometry, and symplectic geometry. Illuminating all these facets of the same object accounts for the length of this paper.

The self-duality equations are equations from gauge theory; geometrically they are defined in terms of connections on principal bundles. While the group of the principal bundle may be chosen arbitrarily for the equations to make sense, we restrict attention here to the simplest case of $SU(2)$ or $SO(3)$. There are two reasons for this. The first, and most obvious, is that it simplifies calculations and avoids the use of inductive processes which are inherent in the consideration of a general Lie group of higher rank. The second reason is that solutions for $SU(2)$ have an intimate relationship with the internal structure of the Riemann surface. As a consequence of results we shall prove about solutions to the self-duality equations, we learn something about the moduli space of complex structures on the surface itself, namely Teichmüller space.

The equations we consider relate a pair of objects: a *connection* A on a principal G -bundle P over the Riemann surface M , and a *Higgs field* Φ . The field Φ is a $(1, 0)$ -form on M with values in the (complex) Lie algebra bundle of P . They may be written as

$$\left. \begin{aligned} d_A''\Phi &= 0, \\ F(A) + [\Phi, \Phi^*] &= 0. \end{aligned} \right\} \quad (*)$$

The first equation says that Φ is holomorphic, and the second is a unitary constraint on the pair.

In the case of $G = \text{SU}(2)$ or $\text{SO}(3)$ a solution to the equations defines a holomorphic rank-2 vector bundle V over M , together with a holomorphic section Φ of $\text{End } V \otimes K$, where K is the canonical bundle of M .

In the first section of the paper we briefly describe the equations and their origin and give examples of solutions, in particular a solution corresponding to a metric of constant negative curvature on the surface. Next, in § 2, we prove a *vanishing theorem* related to solutions of (*). This uses the standard Weitzenböck technique to show that certain holomorphic sections of vector bundles must necessarily be zero in the presence of a solution to the equations. It imposes in particular a constraint on the holomorphic structure of the pair (V, Φ) which we call *stability*. Recall that a vector bundle V is defined to be stable if the degree of any subbundle is less than half the degree of V . Our definition of stability for the pair (V, Φ) , where Φ is a holomorphic section of $\text{End } V \otimes K$, is that any Φ -invariant subbundle must have degree less than half the degree of V . The vanishing theorem shows that a pair arising from a solution of the self-duality equations is necessarily stable.

In § 3 we study this notion of stability from an algebro-geometric point of view concentrating on the question of which vector bundles may occur in a stable pair (V, Φ) . These bundles are classified in terms of their Harder–Narasimhan stratification, but may be more uniformly characterized as those bundles for which a Zariski open set of holomorphic Higgs fields leave invariant no proper subbundle whatsoever. We then consider explicitly the stable pairs for surfaces of low genus: for genus 0 and 1 there are essentially no stable pairs, but for genus 2 we list them. The list is, however, simply a description of individual strata and gives no indication of how they fit together to form a moduli space, which is our main goal.

Having seen that a solution to the self-duality equations gives rise to a stable pair, we prove in § 4 the converse: to each stable pair there exists a solution of the self-duality equations unique modulo unitary gauge transformations. This is a generalization of the theorem of Narasimhan and Seshadri that a stable bundle admits a canonical flat unitary connection. In fact, setting $\Phi = 0$, we obtain their theorem (in the case of rank 2) as a corollary. The proof is modelled on Donaldson's proof of the theorem of Narasimhan and Seshadri. It is an analytical one and makes essential use of Uhlenbeck's weak compactness theorem, one of the most effective tools of gauge theory. Again, like Donaldson's approach, the idea of the proof involves moment maps and symplectic geometry in an infinite-dimensional context. One other corollary of the theorem is the uniformization theorem: every compact Riemann surface of genus $g \geq 2$ admits a metric of constant negative curvature.

The results of §§ 3 and 4 together provide the basis for the rest of the paper. On the one hand, we may use the analysis of § 4 to construct a moduli space for solutions of the self-duality equations and analyse its differential geometric structure, on the other the algebraic geometry of § 3 provides an explicit description of special subspaces and allows us to build up information about the topology and global geometry of the space.

In § 5 we construct the moduli space analytically. The methods are by now quite familiar, having been used in many situations. These involve a vanishing theorem, the Atiyah–Singer index theorem, and Banach space implicit function theorems. The result is that if V is a bundle of odd degree (or equivalently the solution to the self-duality equations is defined on a principal $SO(3)$ bundle with non-zero second Stiefel–Whitney class) then the moduli space of all solutions modulo gauge transformations is a smooth manifold \mathcal{M} of dimension $12(g-1)$ where g is the genus of the Riemann surface.

More recent studies of questions in gauge theory have concentrated not only on the existence and dimension of the moduli space of solutions, but also on the natural Riemannian metric which it carries. The case of magnetic monopoles is one example. In § 6 we study the natural metric on \mathcal{M} . We show first of all that it is complete (a consequence again of Uhlenbeck’s theorem) and secondly that it is a *hyperkähler metric*. This fact is basically a consequence of the structure of the self-duality equations themselves and is due to the fact that, formally speaking, the moduli space \mathcal{M} is a hyperkähler quotient in the sense of [18].

A hyperkähler metric is one which is Kählerian with respect to complex structures $I, J,$ and K which satisfy the algebraic relations of the quaternions i, j, k . Its existence means that \mathcal{M} has many complex structures. The explicit knowledge of \mathcal{M} contained in § 3 when we consider it as the space of equivalence classes of stable pairs describes just one of these structures. As we shall see later, however, holomorphic information about one complex structure yields non-holomorphic information about others.

In § 7 we study the global topology of \mathcal{M} . We show that it is non-compact, connected, and simply-connected and we compute its Betti numbers. To do this we make use of both the differential geometry of § 6 and the algebraic geometry of § 3. We consider the circle action $(A, \Phi) \rightarrow (A, e^{i\theta}\Phi)$ which preserves the self-duality equations and so induces a circle action on \mathcal{M} . This is a group of isometries of the natural metric and preserves the complex structure of \mathcal{M} which corresponds to the equivalence classes of stable pairs (V, Φ) . We then use a method due to Frankel to compute the Betti numbers. The moment map of the circle action with respect to the Kähler form is a perfect Morse function, and hence a study of its critical points, the fixed point sets of the group action, produces formulae for the Betti numbers. The fixed points are analysed by considering the stable pairs (V, Φ) which are fixed (up to equivalence) by the circle action, and these can be explicitly described in terms of symmetric products of the Riemann surface. The final formula (7.6) for the Poincaré polynomial is quite complicated.

If we fix one complex structure of a hyperkähler manifold, then there is a holomorphic (in fact covariant constant) symplectic form. Thus the manifold \mathcal{M} , considered as the moduli space of stable pairs, is in a holomorphic manner a symplectic manifold. We consider it from this point of view in § 8 and show that it may be regarded as an algebraically *completely integrable Hamiltonian system*.

More specifically we consider the map $\det: \mathcal{M} \rightarrow H^0(M; K^2)$ from the $(6g - 6)$ -dimensional complex manifold \mathcal{M} to the $(3g - 3)$ -dimensional space of quadratic differentials defined by taking the determinant of the Higgs field Φ . The analytical estimates of § 4 show that this map is *proper* and general arguments prove that the $3g - 3$ functions which define this map actually commute with respect to the Poisson bracket on \mathcal{M} . From the general point of view of symplectic geometry one expects the generic fibre of the map \det to be a compact complex torus. However, in this case we can prove this directly and show that the fibre over a quadratic differential with simple zeros is the Prym variety of a double covering of M branched over those zeros.

We may go further in studying the symplectic manifold \mathcal{M} . It contains as an open dense set the cotangent bundle of the moduli space \mathcal{N} of stable bundles, with its natural symplectic structure. The fibres of the function \det restricted to $T^*\mathcal{N}$ are non-compact Lagrangian submanifolds (whose compactification in fact generates \mathcal{M}), and we may consider the ‘caustics’ produced by these: the singularities of the projection onto \mathcal{N} . This locus turns out to be the intersection of the theta-divisor of the double covering of M with the Prym variety.

Up to this point we have concentrated on only one complex structure of the hyperkähler family. In § 9 we consider the others. We show that they are all equivalent and give \mathcal{M} the structure of a Stein manifold. This is unlike the previous complex structure of \mathcal{M} , which has compact complex submanifolds.

Adopting the point of view determined by this complex structure on \mathcal{M} we consider the map

$$(A, \Phi) \rightarrow A + \Phi + \Phi^*$$

which associates a complex connection to the pair (A, Φ) . If the self-duality equations for (A, Φ) are satisfied, this connection is flat. Moreover, a vanishing theorem shows that it is irreducible if the solution to the self-duality equations is irreducible.

There is a parallel here with the development of the theory of stable pairs in the first three sections. The analogue of § 4 showing that every stable pair arises from a solution of the self-duality equations is the statement that every irreducible flat connection is gauge-equivalent to a connection of the form $(A + \Phi + \Phi^*)$ for a solution of the self-duality equations. This is indeed true, and is proved by Donaldson in the paper following this. An immediate consequence is that the formula of § 7 for the Betti numbers of the moduli space yields the Betti numbers of a component of the space of irreducible representations of the universal central extension of $\pi_1(M)$ in $SL(2, \mathbb{C})$.

In § 10 we consider one of the bizarre consequences of the existence of a hyperkähler metric on \mathcal{M} . The involution induced by $(A, \Phi) \rightarrow (A, -\Phi)$ is *holomorphic* with respect to the complex structure of stable pairs, but *anti-holomorphic* with respect to one of the other complex structures. It therefore defines a real structure on \mathcal{M} with its Stein manifold complex structure. The map to flat connections described above maps the fixed point sets of the involution onto moduli spaces of real flat connections. Using the explicit knowledge of stable pairs of § 3 we compute those fixed point sets and thus determine information about spaces of flat connections. As a consequence of Donaldson’s theorem there is a completely explicit description of the topology of these spaces: the moduli space of flat $PSL(2, \mathbb{R})$ connections whose associated $\mathbb{R}P^1$ bundle has Euler class

k is diffeomorphic to a complex vector bundle of rank $g - 1 + k$ over the symmetric product $S^{2g-2-k}M$.

The final section deals with one component of the fixed point set of this involution: the one which corresponds to flat bundles of Euler class $(2g - 2)$. In the holomorphic description this is the complex vector space $H^0(M; K^2)$ of quadratic differentials. The self-duality equations (*) lead in this case to the equation $F = -2(1 - \|q\|^2)\omega$ where F is the curvature form of the Kähler metric ω and q is a quadratic differential. (This is formally very similar to the abelian vortex equation in mathematical physics.) The existence theorem of § 4 shows that for any q there is a unique solution. We then show that

$$\hat{h} = q + (\omega + q\bar{q}/\omega) + \bar{q}$$

is a metric of constant negative curvature -4 . Conversely, using the theorem of Earle and Eells in harmonic maps we see that any metric of constant curvature is isometric to one of this form. We thus have a natural diffeomorphism from $\mathbb{C}^{3g-3} = H^0(M; K^2)$ to Teichmüller space, providing a new description of it. It inherits from the hyperkähler metric of the moduli space a complete Kähler metric with a circle action. Neither the complex structure nor the metric structure are those one normally associates with Teichmüller space. This is not surprising as our description has a distinguished base point, the origin in $H^0(M; K^2)$. It is more like an exponential map for Teichmüller space. Nevertheless we show finally that the Kähler form of our metric is the Weil–Petersson symplectic form so that symplectically the two models coincide.

What this paper attempts to do is to give a description of the moduli space of solutions to the self-duality equations and to show the interesting mathematics which is embedded in it. We have not tried to solve explicitly the equations, relying instead on the existence theorem of § 4. Other forms of the self-duality equations have, however, been solved by twistor means and it remains a possibility that this two-dimensional version may also yield to such techniques. We hope to return to this problem in a future paper.

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1. Self-duality

Let A be a connection on a principal G -bundle P over \mathbb{R}^4 , and $F(A)$ its curvature. We write

$$\text{ad}(P) = P \times_G \mathfrak{g}$$

for the vector bundle associated to the adjoint representation, and

$$\Omega^*(\mathbb{R}^4; \text{ad}(P))$$

for the differential forms with values in $\text{ad}(P)$. Then the curvature $F(A) \in \Omega^2(\mathbb{R}^4; \text{ad}(P))$ is a 2-form with values in $\text{ad}(P)$.

A connection is said to satisfy the *self-dual Yang–Mills* equations, or *self-duality equations* for short, if $F(A)$ is invariant under the Hodge star operator:

$$*: \Omega^2(\mathbb{R}^4; \text{ad}(P)) \rightarrow \Omega^2(\mathbb{R}^4; \text{ad}(P)).$$

In terms of a trivialization of P over \mathbb{R}^4 , and the basic coordinates (x_1, x_2, x_3, x_4) , $F(A)$ may be written as a Lie algebra-valued 2-form

$$F(A) = \sum_{i < j} F_{ij} dx_i \wedge dx_j$$

and the self-duality equations then take the form

$$\left. \begin{aligned} F_{12} &= F_{34}, \\ F_{13} &= F_{42}, \\ F_{14} &= F_{23}. \end{aligned} \right\} \quad (1.1)$$

With respect to this trivialization, the *connection* is described by a Lie algebra-valued 1-form

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4$$

and the curvature is then expressed as

$$F(A) = dA + A^2.$$

Alternatively, introducing the covariant derivative

$$\begin{aligned} \nabla_i &= \frac{\partial}{\partial x_i} + A_i, \\ F_{ij} &= \left[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j \right] \\ &= [\nabla_i, \nabla_j], \end{aligned}$$

we now make the assumption that the Lie algebra-valued functions A_i are independent of x_3 and x_4 and hence define functions of $(x_1, x_2) \in \mathbb{R}^2$. Thus A_1 and A_2 define a connection

$$A = A_1 dx_1 + A_2 dx_2$$

over \mathbb{R}^2 , and A_3 and A_4 which we relabel as ϕ_1 and ϕ_2 are auxiliary fields over \mathbb{R}^2 (usually called *Higgs fields*) which are Lie algebra valued.

The self-duality equations (1.1) may now be written as

$$\left. \begin{aligned} F_{12} &= [\nabla_1, \nabla_2] = [\phi_1, \phi_2] = F_{34}, \\ F_{13} &= [\nabla_1, \phi_1] = [\phi_2, \nabla_2] = F_{42}, \\ F_{14} &= [\nabla_1, \phi_2] = [\nabla_2, \phi_1] = F_{23}. \end{aligned} \right\}$$

Introducing the complex Higgs field $\phi = \phi_1 - i\phi_2$ we obtain the two equations

$$\left. \begin{aligned} F &= \frac{1}{2}i[\phi, \phi^*], \\ [\nabla_1 + i\nabla_2, \phi] &= 0. \end{aligned} \right\} \quad (1.2)$$

We assume here that G is the compact real form of a complex Lie group and $*$ is the corresponding anti-involution on the complex Lie algebra. Alternatively, $*$ is the adjoint under some unitary representation.

From a more invariant viewpoint, the equations we have described are the solutions to the self-dual Yang–Mills equations on \mathbb{R}^4 which are invariant under the translation action of the additive group \mathbb{R}^2 :

$$(a_1, a_2) \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 + a_1, x_4 + a_2).$$

The two Higgs fields are then defined by

$$\begin{aligned}\phi_1 &= \nabla_3 - \mathcal{L}_{\partial/\partial x_3}, \\ \phi_2 &= \nabla_4 - \mathcal{L}_{\partial/\partial x_4},\end{aligned}$$

the difference of the covariant derivative and Lie derivative.

From the point of view of the induced connection on the principal bundle P over \mathbb{R}^2 ,

$$F \in \Omega^2(\mathbb{R}^2; \text{ad}(P)) \quad \text{and} \quad \phi \in \Omega^0(\mathbb{R}^2; \text{ad}(P) \otimes \mathbb{C});$$

thus the first of the equations (1.2) is coordinate dependent. However, if we write $z = x_1 + ix_2$ and introduce

$$\Phi = \frac{1}{2}\phi \, dz \in \Omega^{1,0}(\mathbb{R}^2; \text{ad}(P) \otimes \mathbb{C})$$

and

$$\Phi^* = \frac{1}{2}\phi^* \, d\bar{z} \in \Omega^{0,1}(\mathbb{R}^2; \text{ad}(P) \otimes \mathbb{C})$$

then the equations become

$$\left. \begin{aligned} F &= -[\Phi, \Phi^*], \\ d''_A \Phi &= 0, \end{aligned} \right\} \quad (1.3)$$

where $[\Phi, \Phi^*] = \Phi\Phi^* + \Phi^*\Phi$ is the usual extension of the Lie bracket to Lie algebra-valued forms. These are not only coordinate invariant but *conformally invariant*, for a connection A and a $(1, 0)$ form Φ with values in $\text{ad } P$. The second equation of (1.3) simply says that Φ is *holomorphic* with respect to the holomorphic structure on $\text{ad}(P) \otimes_{\mathbb{C}} K$ defined by the connection A on P and the natural holomorphic structure of the canonical bundle K of $(1, 0)$ forms. Since (1.3) is conformally invariant, we may consider solutions on a *compact Riemann surface* M .

To sum up, we consider a connection A on a principal G -bundle P over M (M and G both compact) and a complex Higgs field $\Phi \in \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$. The pair (A, Φ) will be said to satisfy the *self-duality equations* if

$$\left. \begin{aligned} F &= -[\Phi, \Phi^*], \\ d''_A \Phi &= 0. \end{aligned} \right\}$$

REMARK. The formally similar equations

$$\left. \begin{aligned} F &= [\Phi, \Phi^*], \\ d''_A \Phi &= 0, \end{aligned} \right\}$$

which arise in an analogous manner by dimensionally reducing the self-duality equations in \mathbb{R}^4 with signature $(2, 2)$, have a geometrical interpretation: locally, they correspond to harmonic maps from \mathbb{R}^2 to G . We shall deal elsewhere with these equations which, as a consequence of their non-elliptic origins, do not share the same rigidity of structure as do the self-duality equations considered here.

Two examples are of particular geometric interest.

EXAMPLE (1.4). Let $\Phi = 0$; then the self-duality equations are simply $F(A) = 0$ and we are reduced to the consideration of flat unitary connections. This, via the

theorem of Narasimhan and Seshadri [25], is equivalent to the study of stable holomorphic bundles.

EXAMPLE (1.5). Let M be given a Riemannian metric $g = h dz d\bar{z}$ compatible with the conformal structure. Then the Levi-Civita connection is a $U(1)$ connection defined on the canonical bundle K . Let $K^{\frac{1}{2}}$ denote a holomorphic line bundle such that

$$K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$$

with the induced $U(1)$ connection. (These are spinor bundles for the Riemann surface [15].) Let P be the principal $SU(2)$ bundle associated to the rank-2 vector bundle $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ and A the $SU(2)$ connection (reducible to $U(1)$) defined by the Levi-Civita connection. With respect to this decomposition of V , define $\Phi \in \Omega^{1,0}(\text{ad}(P) \otimes \mathbb{C})$ by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \Omega^{1,0}(\text{End } V).$$

Note that this is well-defined since $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})$ is canonically isomorphic to K^{-1} , so 1 denotes the canonical section of $\text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}}) \otimes K$. Note also that Φ is clearly holomorphic. The remaining equation from the self-duality equations (1.3) now takes the form

$$\begin{pmatrix} -\frac{1}{2}F_0 & 0 \\ 0 & \frac{1}{2}F_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h dz d\bar{z},$$

where F_0 is the curvature form of the tangent bundle K^{-1} . Thus the equation becomes

$$F_0 = -2h dz d\bar{z}.$$

In other words, the metric has *constant sectional curvature* -4 .

Solutions of this form are therefore given by isometries of the universal covering of the Riemann surface M onto the upper half-plane.

These two examples are extreme cases of the self-duality equations. Both, however, impose conditions on the underlying holomorphic structure. The holomorphic structure of a vector bundle which admits a flat unitary connection must be *stable* and the genus of a compact Riemann surface with negative curvature must be greater than 1. Both these restrictions may be achieved by vanishing theorems, and in the next section we shall prove vanishing theorems for solutions of the self-duality equations.

2. Vanishing theorems

From now on, we shall restrict attention to the case where $G = SO(3)$. This is by no means necessary for all the arguments, but it makes calculations easier and avoids the use of induction.

There are two cases to consider, depending on whether the second Stiefel-Whitney class $w_2(P)$ is zero or not. If $w_2(P) = 0$, then P is covered by a principal $SU(2)$ -bundle to which we may associate a rank-2 vector bundle V , with $c_1(V) = 0$. If $w_2(P) \neq 0$, then there is a principal $U(2)$ bundle \hat{P} to which P is

associated via the homomorphism $U(2)/Z(U(2)) \cong SO(3)$. Associated to \hat{P} is a rank-2 vector bundle V with $c_1(V)$ odd. Fixing a connection A_0 on $\Lambda^2 V$, we find that a connection A on \hat{P} lifts to one on P , whose curvature is $F(A) + \frac{1}{2}F(A_0)1$.

In both cases, therefore, we are considering a rank-2 vector bundle V , with a fixed connection on the line bundle $\Lambda^2 V$. From a projective embedding of the Riemann surface we know that any line bundle L has a connection A whose curvature is

$$F(A) = (\deg L)\omega,$$

where ω is a positive form. We fix ω and take a corresponding connection on $\Lambda^2 V$.

The vector bundle $\text{ad } P \otimes \mathbb{C}$ over M is in both cases the bundle of endomorphisms of V of trace zero; hence the Higgs field Φ may be thought of as a section of $\text{End}_0 V \otimes K$ where $\text{End}_0 V$ denotes trace-free endomorphisms.

Let $L \subset V$ be a subbundle of rank 1. We shall say that L is Φ -invariant if $\Phi(L) \subseteq L \otimes K$. Now we state the vanishing theorem.

THEOREM (2.1). *Let (A, Φ) satisfy the $SO(3)$ self-duality equations on a compact Riemann surface M and let V be the associated rank-2 complex vector bundle. If $L \subset V$ is a Φ -invariant subbundle, then*

- (i) $\deg(L) \leq \frac{1}{2} \deg(\Lambda^2 V)$, and
- (ii) if equality holds then (A, Φ) reduces to a $U(1)$ solution.

Note. Since $[\Phi, \Phi^*] = 0$ for a $U(1)$ solution, the equations (1.3) simply decouple into a flat connection and a holomorphic 1-form Φ .

Proof. Let $L \subset V$ be a Φ -invariant subbundle and $s \in \Omega^0(M; L^*V)$ be the holomorphic section corresponding to the inclusion. On the line bundle L we put a connection with curvature equal to $(\deg L)\omega$, and using this connection and the connection on V determined by A , we obtain a connection B on L^*V . Since s is holomorphic, we have

$$d''_B s = 0 \in \Omega^{0,1}(M; L^*V). \tag{2.2}$$

Using the hermitian inner product on V , we form $\langle d'_B s, s \rangle \in \Omega^{1,0}(M)$ and then

$$d'' \langle d'_B s, s \rangle = \langle d''_B d'_B s, s \rangle - \langle d'_B s, d'_B s \rangle \in \Omega^{1,1}(M). \tag{2.3}$$

Now

$$d''_B d'_B + d'_B d''_B = F(B)$$

and so, from (2.2) and (2.3),

$$d'' \langle d'_B s, s \rangle = \langle F(B)s, s \rangle - \langle d'_B s, d'_B s \rangle.$$

Integrating over M we obtain by Stokes's theorem

$$\int_M \langle d'_B s, d'_B s \rangle = \int_M \langle F(B)s, s \rangle. \tag{2.4}$$

But now

$$\begin{aligned} F(B)s &= F(A)s - (\deg L)s\omega + \frac{1}{2} \deg(\Lambda^2 V)s\omega \\ &= -[\Phi, \Phi^*]s + (\frac{1}{2} \deg(\Lambda^2 V) - \deg L)s\omega, \end{aligned} \tag{2.5}$$

by the self-duality equations (1.3).

Now if A is a linear transformation and $Av = \lambda v$,

$$\begin{aligned} \langle (AA^* - A^*A)v, v \rangle &= \langle AA^*v, v \rangle - |\lambda|^2 \langle v, v \rangle \\ &= \langle A^*v - \bar{\lambda}v, A^*v - \bar{\lambda}v \rangle \\ &\geq 0 \end{aligned}$$

with equality if and only if $A^*v = \bar{\lambda}v$.

Hence if s is Φ -invariant and $\deg L > \frac{1}{2} \deg(\Lambda^2 V)$, then $\langle F(B)s, s \rangle$ is negative. But from (2.4) its integral is positive; hence we have a contradiction.

If $\deg L = \frac{1}{2} \deg(\Lambda^2 V)$, then $d'_B s = 0$ and s is Φ^* -invariant. Since $\nabla_B s = d'_B s + d''_B s = 0$, the connection on V leaves L invariant. Moreover, Φ and Φ^* leave L invariant. It follows that (A, Φ) reduces to a $U(1)$ solution of the self-duality equations.

REMARKS (2.6). (i) Note that the connection A for a $U(1)$ solution is flat and hence the line bundle has vanishing first Chern class. Consequently, $w_2(P)$ vanishes. Thus principal $SO(3)$ bundles with $w_2(P) \neq 0$ cannot admit solutions to the self-duality equations which reduce to $U(1)$.

(ii) If $\Phi = 0$, then any subbundle is Φ -invariant. The condition $\deg(L) < \frac{1}{2} \deg(\Lambda^2 V)$ for a bundle of rank 2 over a Riemann surface is the notion of *stability* [13]. In the next section we shall examine the algebro-geometric idea of stability for a pair of objects—a rank-2 holomorphic vector bundle V , and a holomorphic section of $\text{End } V \otimes K$ —which Theorem (2.1) suggests.

If g is a section of the bundle of groups $P \times_{\text{Ad}} G$ then g is called a *gauge transformation*. It is an automorphism of the principal bundle which leaves invariant each fibre. The group of all C^∞ gauge transformations acts on connections A and Higgs fields Φ in a natural way and takes one solution of the self-duality equations to another.

The idea of Theorem (2.1) leads to a theorem of uniqueness up to gauge equivalence for solutions of the self-duality equations. This is a consequence of the following more general theorem.

THEOREM (2.7). *Let $(A_1, \Phi_1), (A_2, \Phi_2)$ be two solutions of the self-duality equations on a principal $SO(3)$ bundle over a Riemann surface M . Let V be the associated rank-2 complex vector bundle and assume that there is an isomorphism*

$$h: V \xrightarrow{\cong} V$$

such that

- (i) $d''_{A_2} h = h d''_{A_1}$,
- (ii) $\Phi_2 h = h \Phi_1$.

Then $(A_1, \Phi_1), (A_2, \Phi_2)$ are gauge-equivalent solutions.

Proof. The connections A_1 and A_2 define a connection A on $W = V^* \otimes V$ and Φ_1 and Φ_2 define a Higgs field $\Phi \in \Omega^{1,0}(M; \text{End}(V^* \otimes V))$ by

$$\Phi = \Phi'_1 \otimes 1 + 1 \otimes \Phi_2.$$

Moreover, (A, Φ) satisfies the $SO(4)$ self-duality equations.

Now the vector bundle isomorphism h , thought of as a holomorphic section of W by Condition (i), is Φ -invariant. In fact $\Phi h = 0$ by Condition (ii).

Consequently, as in the proof of (2.1), we obtain

$$\int_M \langle d'_A h, d'_A h \rangle = \int_M -\langle [\Phi, \Phi^*]h, h \rangle = \int_M -\langle \Phi \Phi^* h, h \rangle.$$

Hence $d'_A h = 0$ and $\Phi^* h = 0$. Thus h is covariant constant with respect to the unitary connection A . So therefore is the unitary transformation

$$g = h(h^*h)^{-\frac{1}{2}}.$$

Since $\Phi^* h = 0$, $\Phi_2 h^* = h^* \Phi_1$ as well as $\Phi_2 h = h \Phi_1$. Hence $\Phi_2 g = g \Phi_1$. We therefore have a unitary transformation g such that

$$d_{A_2} g = g d_{A_1}, \quad \Phi_2 g = g \Phi_1.$$

Under the homomorphism

$$U(2) \rightarrow U(2)/Z(U(2)) = SO(3)$$

we obtain an $SO(3)$ gauge transformation taking (A_1, Φ_1) to (A_2, Φ_2) , as required.

3. Stability

We introduce next the generalization of stability of vector bundles which will govern the rest of the paper.

DEFINITION (3.1). Let V be a rank-2 holomorphic vector bundle over a compact Riemann surface M and Φ a holomorphic section of $\text{End } V \otimes K$, where K is the canonical bundle of M . The pair (V, Φ) is defined to be *stable* if, for every Φ -invariant rank-1 subbundle L of V ,

$$\text{deg } L < \frac{1}{2} \text{deg } (\Lambda^2 V).$$

REMARKS (3.2). (i) As mentioned in (2.6), when $\Phi = 0$, this definition reduces to the ordinary definition of stability for a rank-2 vector bundle. However, the pair (V, Φ) may be stable even if $(V, 0)$ is not.

(ii) One example of a stable pair is provided by (1.5). We take $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ on a Riemann surface of genus $g > 1$ and

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H^0(M; \text{End } V \otimes K).$$

Stability is obvious since $K^{-\frac{1}{2}}$, which is of negative degree, is the only Φ -invariant subbundle.

(iii) There are no rank-2 stable pairs on \mathbb{P}^1 , for every vector bundle is of the form

$$V = O(m) \oplus O(n) \quad \text{where } m, n \in \mathbb{Z},$$

and since $K \cong O(-2)$, every $\Phi \in H^0(\mathbb{P}^1; \text{End } V \otimes K)$ is of the form

$$\Phi = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}$$

with $\theta_1 \in H^0(\mathbb{P}^1, O(m-n-2))$ and $\theta_2 \in H^0(\mathbb{P}^1, O(n-m-2))$. Without loss of generality assume that $m-n \geq 0$; hence $\theta_2 = 0$ and $O(m)$ is Φ -invariant. However,

$$\deg O(m) = m \geq \frac{1}{2}(m+n) = \frac{1}{2} \deg(\Lambda^2 V).$$

(iv) On an elliptic curve, every indecomposable bundle is, after tensoring with a line bundle, equivalent to the non-trivial extension [3]

$$0 \rightarrow O \rightarrow V \rightarrow O \rightarrow 0$$

defined by $H^1(M; O) \cong \mathbb{C}$ or

$$0 \rightarrow O \rightarrow V \rightarrow O(1) \rightarrow 0$$

where $O(1)$ is a line bundle of degree 1, defined by $H^1(M; O(-1)) \cong \mathbb{C}$. Since the canonical bundle is trivial, the Higgs fields Φ are endomorphisms of V , but in the first case the distinguished trivial subbundle $L \cong O$ is invariant by each endomorphism. But $\deg L = 0 = \frac{1}{2} \deg \Lambda^2 V$. The second example is stable, and hence the only endomorphisms are scalars. Thus $(V, 0)$ is the only stable pair.

For a decomposable bundle $L_1 \oplus L_2$ each Φ is of the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where $b \in H^0(M; L_2^* L_1)$ and $c \in H^0(M; L_1^* L_2)$. If $\deg(L_2^* L_1) \geq 0$ and $L_2^* L_1$ is non-trivial, $c = 0$ and then L_1 is invariant but $\deg L_1 \geq \frac{1}{2} \deg(L_1 L_2)$. If $L_1 \cong L_2$, then a, b, c are constants and the matrix certainly has an eigenspace which is a subbundle of degree zero.

Consequently, there are no stable pairs for an elliptic curve other than $(V, 0)$ where V is the unique non-trivial bundle with odd degree.

In the case of surfaces of genus greater than 1, stable pairs occur with more frequency, but there are still restrictions on the holomorphic structure of the underlying vector bundle V . We recall the various types of rank-2 bundles on a Riemann surface (see [5]):

- (i) V is stable if for each subbundle L , $\deg L < \frac{1}{2} \deg(\Lambda^2 V)$;
- (ii) V is semi-stable if for each subbundle L , $\deg L \leq \frac{1}{2} \deg(\Lambda^2 V)$;
- (iii) if V is not semi-stable, there is a unique subbundle L_V with $\deg L_V > \frac{1}{2} \deg(\Lambda^2 V)$.

The following proposition catalogues the types which can occur in stable pairs.

PROPOSITION (3.3). *Let M be a compact Riemann surface of genus $g > 1$. A rank-2 vector bundle V occurs in a stable pair (V, Φ) if and only if one of the following holds:*

- (i) V is stable;
- (ii) V is semi-stable and $g > 2$;
- (iii) if V is semi-stable and $g = 2$ then $V \cong U \otimes L$ where U is either decomposable or an extension of the trivial bundle by itself;
- (iv) V is not semi-stable and $\dim H^0(M; L_V^{-2} K \otimes \Lambda^2 V)$ is greater than 1, where L_V is the canonical subbundle;

(v) V is decomposable as

$$V = L_V \oplus (L_V^* \otimes \Lambda^2 V) \quad \text{and} \quad \dim H^0(M; L_V^{-2} K \otimes \Lambda^2 V) = 1.$$

The method of proof of (3.3) leads to the following more uniform characterization which will be useful in proving the analytical results of § 4.

PROPOSITION (3.4). *Let M be a compact Riemann surface of genus $g > 1$. A rank-2 vector bundle V occurs in a stable pair (V, Φ) if and only if there is a Zariski open subset $U \subseteq H^0(M; \text{End } V \otimes K)$ such that if $\Phi \in U$, then Φ leaves invariant no proper subbundle.*

Proof of (3.3). We shall use the complex surface $P(V)$ obtained by projectivizing the vector bundle V , and convert questions concerning sections of vector bundles on M to questions of divisors on $P(V)$. There is a tautological bundle H on $P(V)$ (positive on the fibres) whose sections along the fibre over $x \in M$ constitute the vector space V_x^* . Thus, if $p: P(V) \rightarrow M$ denotes the projection, $p_* H = V^*$ where $p_* H$ is the direct image sheaf, and

$$p_* H^2 = S^2 V^*$$

where $S^2 V^*$ is the second symmetric power of V^* , the bundle of quadratic forms on V . If $A \in \text{End } V$, then $Av \wedge v$ defines a quadratic map from V to $\Lambda^2 V$, the scalar endomorphisms going to zero. Using this, we may identify as vector bundles

$$S^2 V^* \otimes \Lambda^2 V \cong \text{End}_0 V,$$

where $\text{End}_0 V$ denotes the traceless endomorphisms, and consequently

$$p_* H^2 K \otimes \Lambda^2 V \cong \text{End}_0 V \otimes K.$$

There is, moreover, an isomorphism of sections,

$$s: H^0(M; \text{End}_0 V \otimes K) \xrightarrow{\cong} H^0(P(V); H^2 K \otimes \Lambda^2 V) \quad (3.5)$$

obtained by pulling back and pushing down.

In this framework, a subbundle $L \subset V$ defines a section of $L^* V$ and hence a section on $P(V)$ of $HL^* \otimes \Lambda^2 V$. Its divisor $D(L)$ is the canonical section of $P(V) \rightarrow M$ determined by the rank-1 subbundle. Moreover, $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ leaves L invariant if and only if

$$\Phi v \wedge v = 0 \quad \text{for all } v \in L,$$

that is, if $s(\Phi) \in H^0(P(V), H^2 K \otimes \Lambda^2 V)$ vanishes on $D(L)$. In this case $D(L)$ is a proper component of the divisor of $s(\Phi)$. If the divisor of $s(\Phi)$ were irreducible, then Φ would leave invariant no subbundle. It is through irreducibility we shall prove the proposition, making use of Bertini's theorem: the generic divisor of a linear system of dimension at least 2 with no fixed component is irreducible.

First we check that the linear system of divisors of $H^2 K \otimes \Lambda^2 V$ has dimension at least 2. This is equivalent from (3.5) to $\dim H^0(M; \text{End}_0 V \otimes K) \geq 3$. However, by the Riemann–Roch theorem,

$$\dim H^0(M; \text{End}_0 V \otimes K) - \dim H^1(M; \text{End}_0 V \otimes K) = 3g - 3 \geq 3 \quad \text{if } g > 1. \quad (3.6)$$

Now suppose $H^2K \otimes \Lambda^2V$ has no fixed components. Then, by Bertini's theorem, a generic $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ leaves invariant no subbundle, and so in particular (V, Φ) is a stable pair. If the system has a fixed part, it is contained in the divisors of a line bundle of one of the following types:

$$(a) \ H^2L, \quad (b) \ HL, \quad (c) \ L,$$

where L is a bundle pulled back from M .

For Type (c) the divisors of the bundle $H^2KL^* \otimes \Lambda^2V$ have no fixed part, so by Bertini a generic $\Phi \in H^0(M; \text{End}_0 V \otimes KL^*)$ leaves invariant no subbundle. Multiplying by the fixed section of L on M , we find that neither does a generic section of $\text{End}_0 V \otimes K$.

For Type (a), every $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ is of the form $\Phi = \Phi_0 s$, where $\Phi_0 \in H^0(M; \text{End}_0 V \otimes L\Lambda^2V^*)$ and $s \in H^0(M; L^*K \otimes \Lambda^2V)$.

If $\text{Tr } \Phi_0^2 \neq 0$, then consider the linear map

$$\alpha: H^0(M; L^*K \otimes \Lambda^2V) \rightarrow H^0(M; K^2)$$

defined by

$$\alpha(s) = \text{Tr } \Phi_0^2 s_1 s$$

for some fixed $s_1 \in H^0(M; L^*K \otimes \Lambda^2V)$. This is clearly injective, but from (3.6),

$$\dim H^0(M; L^*K \otimes \Lambda^2V) \geq 3g - 3 = \dim H^0(M; K^2),$$

so α is an isomorphism. However, $\alpha(s)$ vanishes at the zeros of s_1 but K^2 has no basepoints. Thus $\text{Tr } \Phi_0^2 = \det \Phi_0 = 0$. The kernel of Φ_0 then defines a subbundle $L \subset V$ invariant by all $\Phi \in H^0(M; \text{End}_0 V \otimes K)$.

In Type (b) the subbundle defined by the fixed section of HL is by definition invariant for all Φ .

We conclude therefore that the vector bundles V for which a generic Φ leaves no subbundle invariant are those which have no subbundle invariant by all Φ .

To return to the statement of Proposition (3.3), if V is stable, then clearly (V, Φ) is stable for all Φ .

Assume therefore that V has a subbundle L with $\deg L \geq \frac{1}{2} \deg \Lambda^2V$. Then V is an extension

$$0 \rightarrow L \rightarrow V \rightarrow L^* \otimes \Lambda^2V \rightarrow 0$$

and there is a corresponding subbundle

$$KL^2 \otimes \Lambda^2V^* \subset \text{End}_0 V \otimes K$$

which leaves only L invariant. Since

$$\deg(KL^2 \otimes \Lambda^2V^*) \geq 2g - 2,$$

this subbundle has sections. If V has a subbundle invariant by all Φ , this must clearly be it. We investigate the possibilities by considering the exact sequence of vector bundles

$$0 \rightarrow V^* \otimes KL \rightarrow \text{End}_0 V \otimes K \rightarrow L^{-2}K \otimes \Lambda^2V \rightarrow 0. \quad (3.7)$$

Sections of $V^* \otimes KL$ are the sections $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ which leave L invariant, so from the exact cohomology sequence of (3.7) we must have the

coboundary map injective:

$$H^0(M; L^{-2}K \otimes \Lambda^2 V) \xrightarrow{\delta} H^1(M; V^* \otimes KL). \quad (3.8)$$

Consider the exact sequence of vector bundles

$$0 \longrightarrow KL^2 \otimes \Lambda^2 V^* \longrightarrow V^* \otimes KL \xrightarrow{\pi} K \longrightarrow 0 \quad (3.9)$$

and its cohomology sequence.

If $\deg L > \frac{1}{2} \deg \Lambda^2 V$, then $H^1(M; KL^2 \otimes \Lambda^2 V^*) = 0$, so

$$H^1(M; V^* \otimes KL) \cong H^1(M; K) \cong \mathbb{C}.$$

Thus if $\dim H^0(M; L^{-2}K \otimes \Lambda^2 V) \geq 2$, then δ can never be injective, and we have a contradiction. This gives Case (iv).

The map

$$\pi\delta: H^0(M; L^{-2}K \otimes \Lambda^2 V) \rightarrow H^1(M; K)$$

is given by the product with the extension class $e \in H^1(M; L^2 \otimes \Lambda^2 V^*)$ defining V . By Serre duality, this is surjective if $e \neq 0$. This provides Case (v).

Clearly if $H^0(M; L^{-2}K \otimes \Lambda^2 V) = 0$, then δ is injective.

It remains to consider the semi-stable situation $\deg L = \frac{1}{2} \deg \Lambda^2 V$.

If $L^2 \otimes \Lambda^2 V^*$ is non-trivial, $H^1(M; KL^2 \otimes \Lambda^2 V^*) = 0$ and so

$$H^1(M; V^* \otimes KL) \cong H^1(M; K) \cong \mathbb{C}$$

as above. Since then

$$\dim H^0(M; L^{-2}K \otimes \Lambda^2 V) = g - 1,$$

δ can be injective only if $g = 2$. By Serre duality again this means that the extension defining V is non-trivial.

If $L^2 \otimes \Lambda^2 V^*$ is trivial, then the coboundary map

$$H^0(M; K) \rightarrow H^1(M; K)$$

in (3.9) is surjective by Serre duality for a non-trivial extension, and hence $H^1(M; V^* \otimes KL)$ is still 1-dimensional. However, in this case

$$\dim H^0(M; L^{-2} \otimes \Lambda^2 V) = g$$

and δ can never be injective if $g \geq 2$. The only possibility is the trivial extension

$$V = L \oplus (L^* \otimes \Lambda^2 V) = L \oplus (O \oplus O),$$

but then $\delta \equiv 0$, so this occurs in a stable pair. This finally deals with Case (iii).

We have shown that all Cases (i)–(v) occur in stable pairs. To prove the converse we simply look at the excluded bundles, and these were characterized as unstable bundles with a subbundle L invariant by all $\Phi \in H^0(M; \text{End}_0 V \otimes K)$. We saw moreover that $\deg L \geq \frac{1}{2} \deg \Lambda^2 V$ and so there are no stable pairs (V, Φ) .

Proof of (3.4). In the course of the proof of Proposition (3.3) we established Proposition (3.4) for all bundles except stable bundles, and what is required to

complete the proof is to show that a stable bundle V has no subbundle L invariant by all $\Phi \in H^0(M; \text{End } V \otimes K)$.

Suppose there were such a bundle, so that V is an extension

$$0 \rightarrow L \rightarrow V \rightarrow L^* \otimes \Lambda^2 V \rightarrow 0.$$

Since V is stable,

$$\deg L < \frac{1}{2} \deg \Lambda^2 V.$$

If L is invariant by all $\Phi \in H^0(M; \text{End } V \otimes K)$ then from (3.7) all such Φ are sections of $V^* \otimes KL$. Now consider the exact sequence (3.9),

$$0 \rightarrow KL^2 \otimes \Lambda^2 V^* \rightarrow V^* \otimes KL \rightarrow K \rightarrow 0.$$

Since $\deg(KL^2 \otimes \Lambda^2 V^*) < 2g - 2$, we have

$$\dim H^0(M; KL^2 \otimes \Lambda^2 V^*) < g. \quad (3.10)$$

But since $\dim H^0(M; K) = g$, the exact cohomology sequence for (3.9) gives

$$\dim H^0(M; V^* \otimes KL) \leq 2g - 1.$$

On the other hand, from the Riemann–Roch theorem,

$$\dim H^0(M; \text{End } V_0 \otimes K) \geq 3g - 3. \quad (3.11)$$

Thus if $H^0(M; \text{End}_0 V \otimes K) \cong H^0(M; V^* \otimes KL)$, we have necessarily $g = 2$, and in (3.10),

$$\dim H^0(M; KL^2 \otimes \Lambda^2 V^*) = 1.$$

Since $\deg K = 2$ and $\deg L^2 \otimes \Lambda^2 V^* < 0$, there are only two possibilities:

$$L^2 \otimes \Lambda^2 V^* \cong K^{-1} \quad \text{or} \quad L^2 \otimes \Lambda^2 V^* \cong K^{-1}(x)$$

for some point $x \in M$.

In the first case the exact sequence (3.9) is

$$0 \rightarrow O \rightarrow V^* \otimes KL \rightarrow K \rightarrow 0$$

and since V is indecomposable this is a non-trivial extension. But the coboundary map

$$\delta: H^0(M; K) \rightarrow H^1(M; O)$$

is non-zero, so that $\dim H^0(M; V^* \otimes KL) \leq 2$, which contradicts (3.11).

In the second case, the exact sequence is

$$0 \rightarrow O(x) \rightarrow V^* \otimes KL \rightarrow K \rightarrow 0.$$

The coboundary map

$$\delta: H^0(M; K) \rightarrow H^1(M; O(x))$$

is given by the cup product with the extension class in $H^1(M; K^{-1}(x))$. Dualizing, we see that the map δ is non-zero if the product map

$$H^0(M; K) \otimes H^0(M; K(-x)) \rightarrow H^0(M; K^2(-x))$$

is surjective. This however is clearly true since $K(-x) \cong O(x')$ and $\dim H^0(M; K) = \dim H^0(M; K^2(-x)) = 2$.

Thus again we obtain $\dim H^0(M; V^* \otimes KL) \leq 2$ and a contradiction.

REMARKS (3.12). Conditions (iv) and (v) of Proposition (3.3) show that when a bundle V is unstable, there is a constraint on the canonical subbundle L_V in order for it to belong to a stable pair. In particular, since $\dim H^0(M ; L_V^{-2}K \otimes \Lambda^2V) \geq 1$, we must have firstly

$$0 < \deg(L_V^2 \otimes \Lambda^2V^*) \leq 2g - 2$$

and then, if $\deg(L_V^2 \otimes \Lambda^2V^*) > g - 2$, the line bundle must be special.

Note that if $\deg(L_V^2 \otimes \Lambda^2V^*) = 2g - 2$, then the condition

$$\dim H^0(M ; L_V^{-2}K \otimes \Lambda^2V) = 1$$

implies that

$$L_V^2 \otimes \Lambda^2V^* \cong K.$$

From Case (v) of (3.3) this means that

$$V \cong (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes L_V K^{-\frac{1}{2}},$$

which is equivalent to the bundle occurring in Example (3.2)(ii).

The information we have derived so far is enough to provide an explicit description of stable pairs for a surface of genus 2. We split the problem into two cases corresponding to $\deg(\Lambda^2V)$ even or odd. By tensoring with a line bundle we may assume that

$$\Lambda^2V \cong O \quad \text{or} \quad \Lambda^2V \cong O(x)$$

for some fixed point $x \in M$.

EXAMPLE (3.13). If $g = 2$ and $\Lambda^2V \cong O$ then (3.3) and Remark (3.12) yield the following possibilities for V :

- (i) V is stable;
- (ii) $V \cong L \oplus L^*$ where L^2 is non-trivial;
- (iii) $V \cong L \oplus L^* \cong L \otimes (O \oplus O)$ where L^2 is trivial;
- (iv) V is a non-trivial extension of the trivial bundle O by itself;
- (v) $V \cong K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ where $K^{\frac{1}{2}}$ is a holomorphic line bundle such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$.

In Case (i), of course, (V, Φ) is stable for any Φ in the 3-dimensional space $H^0(M ; \text{End}_0 V \otimes K)$. The stable bundle V itself is, from a result of Narasimhan and Ramanan [26], determined by the subbundles of $\deg -1$ it contains. This is a divisor of the system 2Θ in the Jacobian $J^1(M)$, and so each stable bundle is determined by a point in the 3-dimensional projective space $P(H^0(J^1(M) ; 2\Theta))$. In fact the stable ones correspond bijectively to the complement of a Kummer surface in the projective space, parametrizing the bundles which are decomposable into a sum of line bundles of degree zero.

For Case (ii), (iii), and (v), any $\Phi \in H^0(M ; \text{End}_0 V \otimes K)$, where V is decomposable as $V = L \oplus L^*$, may be expressed as

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where $a \in H^0(M ; K)$, $b \in H^0(M ; L^2K)$, and $c \in H^0(M ; L^{-2}K)$.

In Case (ii), it is trivial that the only subbundles with degree greater than or equal to $\frac{1}{2} \deg \Lambda^2 V = 0$ are L and L^* . Hence (V, Φ) is stable if and only if neither b nor c is identically zero.

Since $\dim H^0(M; L^2 K) = \dim H^0(M; L^{-2} K) = 1$, Φ is stable if it lies in the complement of the two hyperplanes $b = 0$ and $c = 0$ in the 4-dimensional space $H^0(M; \text{End}_0 V \otimes K)$.

In Case (v), $K^{-\frac{1}{2}}$ is the only subbundle with degree at least zero. In this case

$$\begin{aligned} \dim H^0(M; (K^{-\frac{1}{2}})^2 K) &= \dim H^0(M; O) = 1, \\ \dim H^0(M; (K^{\frac{1}{2}})^2 K) &= \dim H^0(M; K^2) = 3, \end{aligned}$$

and Φ is stable if $c \neq 0$. This is the complement of a hyperplane in the 6-dimensional space $H^0(M; \text{End}_0 V \otimes K)$.

In Case (iii), $V \cong L \otimes \mathbb{C}^2$ and the subbundles of degree at least zero are simply the fixed subspaces of \mathbb{C}^2 . We can express Φ uniquely as

$$\Phi = A_1 \alpha_1 + A_2 \alpha_2$$

where the A_i are 2×2 traceless matrices, and α_1, α_2 is a basis of $H^0(M; K)$. The subspaces of the 6-dimensional space $H^0(M; \text{End}_0 V \otimes K)$ of stable Φ is then isomorphic to the set of all pairs (A_1, A_2) of matrices with no common eigenspace.

In Case (iv), the trivial subbundle L of V is unique. Hence, from the proof of (3.3), the kernel of δ in (3.8) is 1-dimensional, so from the exact sequence of (3.7), $\dim H^0(M; \text{End}_0 V \otimes K) = 4$ and the stable Φ lie in the complement of the hyperplane of sections which leave L invariant.

EXAMPLE (3.14). If $g = 2$ and $\Lambda^2 V \cong O(x)$, then we have the following possibilities from Proposition (3.3) and Remark (3.12):

- (i) V is stable;
- (ii) $V \cong L \oplus L^*(x)$ where L is a line bundle of degree 1, with $\dim H^0(M; KL^{-2}(x)) = 1$.

In Case (i), (V, Φ) is again stable for any Φ in the 3-dimensional space $H^0(M; \text{End}_0 V \otimes K)$. The stable bundles considered here are parametrized by the intersection of two quadrics in P^5 (see [26, 28]). Each point $y \in M$, considered as a double covering of the projective line, parametrizes a quadric Q_y of the pencil together with a choice of α -plane or β -plane. Fixing a point q in the intersection of the quadrics of the pencil, we define the fibre of $P(V)$ over y to consist of the α -planes in Q_y passing through q .

In Case (ii), $KL^{-2}(x)$ is a line bundle of degree 1, so the condition that $\dim H^0(M; KL^{-2}(x)) = 1$ means that

$$KL^{-2}(x) \cong O(y)$$

for some $y \in M$. Modulo tensoring with a line bundle of order 2, the relevant vector bundles are thus parametrized by the points of M itself.

The Higgs field Φ again has the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

where $a \in H^0(M; K)$, $b \in H^0(M; KL^2(-x))$, and $c \in H^0(M; KL^{-2}(x))$. The only

subbundle with degree greater than $\frac{1}{2} \deg(\Lambda^2 V) = \frac{1}{2}$ is L itself, so Φ is stable if $c \neq 0$. Thus the stable Φ lie in the complement of a hyperplane in the 4-dimensional space $H^0(M; \text{End}_0 V \otimes K)$.

Note throughout these examples that the simpler the description of V , the more complicated is the stability condition on Φ . Treating the complex structure on V and the constraint on Φ as separate objects clearly leads to a complex description of stable pairs (V, Φ) . The aim of this paper is to put the two together and describe equivalence classes in terms of a moduli space which is endowed with a uniform geometrical structure.

There is a further condition which a stable pair (V, Φ) possesses, analogous to the statement that a stable bundle is simple:

PROPOSITION (3.15). *Let (V_1, Φ_1) and (V_2, Φ_2) be stable pairs with $\Lambda^2 V_1 \cong \Lambda^2 V_2$ and $\Psi: V_1 \rightarrow V_2$ a non-zero homomorphism such that $\Psi\Phi_1 = \Phi_2\Psi$. Then Ψ is an isomorphism. If $(V_1, \Phi_1) = (V_2, \Phi_2)$, then Ψ is a scalar multiplication.*

Proof. If Ψ is not an isomorphism then there are subbundles $L_1 \subset V_1$ and $L_2 \subset V_2$ such that $\text{im } \Psi \subseteq L_2$ and $L_1 \subseteq \ker \Psi$ which, since $\Psi\Phi_1 = \Phi_2\Psi$, are invariant by Φ_1 and Φ_2 respectively. Hence by stability,

$$\deg L_1 < \frac{1}{2} \deg \Lambda^2 V_1 = \frac{1}{2}d$$

and

$$\deg L_2 < \frac{1}{2} \deg \Lambda^2 V_2 = \frac{1}{2}d.$$

But Ψ defines a non-zero homomorphism from V_1/L_1 to L_2 , and hence

$$\deg(V_1/L_1) \leq \deg L_2 < \frac{1}{2}d.$$

But $\deg(V_1/L_1) = \deg \Lambda^2 V_1 - \deg L_1 > \frac{1}{2}d$, which is a contradiction. Hence Ψ must be an isomorphism.

If $(V_1, \Phi_1) = (V_2, \Phi_2)$, then $\det \Psi$ is a constant and so the eigenvalues of Ψ are constant.

Thus if we have distinct eigenvalues, the eigenspaces decompose V as a direct sum:

$$V \cong L_1 \oplus L_2.$$

Since $\Phi_1 = \Phi_2$ leaves L_1 and L_2 invariant, we have by stability

$$\deg L_1 < \frac{1}{2} \deg \Lambda^2 V = \frac{1}{2}(\deg L_1 + \deg L_2),$$

$$\deg L_2 < \frac{1}{2} \deg \Lambda^2 V = \frac{1}{2}(\deg L_1 + \deg L_2),$$

which is clearly a contradiction as before.

If $\Psi = \lambda I + \Psi_0$ where Ψ_0 is nilpotent, then $\ker \Psi_0$ defines a subbundle invariant by Φ . As in (3.9) this means that Ψ_0 defines a section of $L^2 \otimes \Lambda^2 V^*$, but by stability $\deg(L^2 \otimes \Lambda^2 V^*) < 0$, so all sections vanish. Thus $\Psi_0 = 0$ and Ψ is a scalar.

4. An existence theorem

The link between the algebro-geometrical idea of stability and the self-duality equations is provided by Theorem (2.1). It implies that if (A, Φ) is an irreducible

solution to the $SO(3)$ self-duality equations on a compact Riemann surface M , then the associated pair (V, Φ) is stable. The aim of this section is to prove the converse: to each stable pair (V, Φ) there corresponds a solution to the $SO(3)$ self-duality equations, unique up to gauge equivalence. Using this result we shall be able to knit together the equivalence classes of stable pairs (V, Φ) into a differentiable manifold with a structure which is rich from the complex, Riemannian and symplectic point of view. Note that if $\Phi = 0$, the converse result is equivalent to the theorem of Narasimhan and Seshadri that a stable bundle admits a canonical flat unitary connection [25]. In fact our proof is modelled on Donaldson's proof of this theorem [6]. It is an analytical proof which makes use of one non-trivial and highly effective tool in gauge theory: Uhlenbeck's weak compactness theorem [32].

To motivate the proof, we consider the structure of the equation

$$F + [\Phi, \Phi^*] = 0 \quad (4.1)$$

which forms part of (1.3), in terms of *moment maps*.

Recall that if N is a Kähler manifold with Kähler form ω , and X is a Killing field which preserves ω , then

$$0 = \mathcal{L}_X \omega = d(i(X)\omega) + i(X) d\omega = d(i(X)\omega),$$

so that if $H^1(N; \mathbb{R}) = 0$ then

$$i(X)\omega = df_X$$

for some function f , the Hamiltonian function for the vector field X . Moreover,

$$g(\text{grad } f_X, Y) = df_X(Y) = \omega(X, Y) = g(IX, Y),$$

so that

$$\text{grad } f_X = IX. \quad (4.2)$$

If a Lie group G acts on N by isometries which preserve ω , then under rather general conditions the functions f_X fit together to give an equivariant moment map $\mu: N \rightarrow \mathfrak{g}^*$ defined by

$$\langle \mu(x), X \rangle = f_X(x),$$

where we identify the vector field X with the corresponding element in the Lie algebra \mathfrak{g} of G .

There are two fundamental examples. The first is $N = \text{End } \mathbb{C}^n$ with Kähler metric

$$g(A, B) = \text{Re Tr}(AB^*)$$

and $G = U(n)$ acting by conjugation. Then if $X \in \mathfrak{g}$, and

$$f_X(A) = \frac{1}{2}i \text{Tr}([A, A^*]X),$$

we have

$$\begin{aligned} df_X(Y) &= \frac{1}{2}i \text{Tr}([Y, A^*]X + [A, Y^*]X) \\ &= i \text{Im Tr}([X, A]Y^*) \\ &= \text{Re Tr}([iX, A]Y^*). \end{aligned}$$

Hence $\mu(A) = \frac{1}{2}i[A, A^*]$ is a moment map.

The second example is infinite-dimensional and due to Atiyah and Bott [5]. Take N to be the infinite-dimensional affine space \mathcal{A} of connections on a unitary principal bundle P . Each such connection is determined uniquely by its $(0, 1)$ part d''_A and the tangent space to \mathcal{A} at A is the complex space $\Omega^{0,1}(M; \text{ad } P \otimes \mathbb{C})$, which carries the Kähler metric

$$g(\Psi, \Psi) = 2i \int_M \text{Tr}(\Psi^* \Psi).$$

The moment map for the action of the group of gauge transformations \mathcal{G} is then

$$\mu_1(A) = F(A) \in \Omega^2(M; \text{ad } P).$$

If we consider the natural action of \mathcal{G} on $\Omega^{1,0}(\text{ad } P \otimes \mathbb{C})$, then it is easy to see, from the first example, that with the Kähler metric

$$g(\Phi, \Phi) = 2i \int_M \text{Tr}(\Phi \Phi^*),$$

the moment map is

$$\mu_2(\Phi) = [\Phi, \Phi^*].$$

Thus the equation (4.1) is equivalent to the equation

$$\mu(A, \Phi) = \mu_1(A) + \mu_2(\Phi) = 0$$

for the moment map of the group of gauge transformations \mathcal{G} acting on both factors of the infinite-dimensional Kähler manifold

$$N = \mathcal{A} \times \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C}). \tag{4.2}$$

Now suppose the Lie algebra \mathfrak{g} has an invariant inner product and the action of the group G extends to a complex group G^c of holomorphic transformations. We restrict the function $\|\mu\|^2$ to an orbit and look for the critical points.

If G^c acts freely (or with finite isotropy) on the orbit, since $\text{grad } \mu_x = IX$, we have that

$$\text{grad}_x \mu \in \text{Hom}(\mathfrak{g}, T_x)$$

is *injective*. Thus

$$\text{grad } \|\mu\|^2 = 2(\mu, \text{grad } \mu) = 0 \quad \text{at } x \in N$$

only if $\mu(x) = 0$.

From this point of view we attempt to solve the equation $F + [\Phi, \Phi^*] = 0$ by considering an orbit of a stable pair

$$(A, \Phi) \in \mathcal{A} \times \Omega^{1,0}(M, \text{End}_0 V)$$

under the group of complex gauge transformations $\mathcal{G}^c = \Omega^0(M; \text{Aut}_0 V)$, the group of automorphisms of V with determinant 1. Choosing a metric on the Riemann surface M , we find a minimum for

$$\|\mu\|^2 = \int_M \|F + [\Phi, \Phi^*]\|^2$$

on the orbit. By Proposition (3.15), \mathcal{G}^c acts freely on a stable orbit, so we will

produce a solution of

$$\mu = F + [\Phi, \Phi^*] = 0.$$

THEOREM (4.3). *Let A be an $\text{SO}(3)$ connection on a bundle P over a compact Riemann surface M of genus $g > 1$, and let $\Phi \in \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$ satisfy $d_A'' \Phi = 0$. Let V be an associated rank-2 vector bundle with complex structure determined by A . If (V, Φ) is a stable pair, then there exists an automorphism of V of determinant 1, unique modulo $\text{SO}(3)$ gauge transformations, which takes (A, Φ) to a solution of the equation $F(A) + [\Phi, \Phi^*] = 0$.*

Proof. As in [27, 6] we shall work with connections which differ from a smooth connection by an element of the Sobolev space L_1^2 , and use automorphisms which lie in L_2^2 . Since (from [5]) every L_2^2 orbit in the L_1^2 space of connections contains a C^∞ connection there is no loss of generality as far as A is concerned. Also, since Φ satisfies the elliptic equation $d_A'' \Phi = 0$, we can by elliptic regularity deduce that Φ is C^∞ . We therefore start with Φ in the L_1^2 space, so that $[\Phi, \Phi^*] \in L^2$.

We have fixed a metric on M , and as discussed above, consider the functional

$$f(A, \Phi) = \int_M \|F(A) + [\Phi, \Phi^*]\|^2$$

on an orbit under the group of L_2^2 complex gauge transformations of a stable pair (A, Φ) . Let (A_n, Φ_n) be a minimizing sequence for f on this orbit; hence, in particular,

$$\|F(A_n) + [\Phi_n, \Phi_n^*]\|_{L^2} < m. \quad (4.4)$$

We shall use the theorem of Uhlenbeck [32] which implies that if A_n is a sequence of L_1^2 connections over M for which $F(A_n)$ is bounded in L^2 , then there are unitary gauge transformations g_n for which $g_n \cdot A_n$ has a weakly convergent subsequence.

The inequality (4.4) does not immediately give an L^2 bound on $F(A_n)$. However, we may use the Weitzenböck formula (2.4) applied to $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ to obtain

$$\begin{aligned} \int_M \langle d_B' \Phi, d_B' \Phi \rangle &= \int_M \langle F(B) \Phi, \Phi \rangle \\ &= \int_M \langle [F(A), \Phi], \Phi \rangle + (2g - 2) \int_M \omega \langle \Phi, \Phi \rangle \\ &= \int_M \langle F(A), [\Phi, \Phi^*] \rangle + (2g - 2) \int_M \omega \langle \Phi, \Phi \rangle, \end{aligned} \quad (4.5)$$

where $(2g - 2)\omega$ is the positive curvature of a connection on the canonical bundle out of which the connection B on $\text{End}_0 V \otimes K$ is formed.

We deduce the inequality

$$0 \leq \langle F, [\Phi, \Phi^*] \rangle_{L^2} + c \|\Phi\|_{L^2}^2. \quad (4.6)$$

Hence

$$0 \leq \langle F + [\Phi, \Phi^*], [\Phi, \Phi^*] \rangle_{L^2} - \|[\Phi, \Phi^*]\|_{L^2}^2 + c \|\Phi\|_{L^2}^2 \quad (4.7)$$

and, from (4.4),

$$\langle F_n + [\Phi_n, \Phi_n^*], [\Phi_n, \Phi_n^*] \rangle_{L^2} \leq m \|[\Phi_n, \Phi_n^*]\|_{L^2}. \quad (4.8)$$

Putting together (4.7) and (4.8) we derive the inequality

$$\|[\Phi_n, \Phi_n^*]\|_{L^2}^2 \leq c_1 + c_2 \|\Phi_n\|_{L^2}^2 \quad (4.9)$$

for positive constants c_1 and c_2 .

Now if A is a 2×2 matrix of trace zero,

$$\begin{aligned} \text{Tr}(AA^* - A^*A)^2 &= 2 \text{Tr}(AA^*)^2 - 2 \text{Tr} A^2 A^{*2} \\ &= 2 \text{Tr}(AA^*)^2 - 4 |\det A|^2. \end{aligned}$$

Since $\text{Tr}(AA^*)^2 \geq \frac{1}{2}(\text{Tr} AA^*)^2$, using $\|A\|^2 = \text{Tr} AA^*$, we have the inequality

$$\|[A, A^*]\|^2 + 4 |\det A|^2 \geq \|A\|^4. \quad (4.10)$$

Now $L_1^2 \subset L^4$ and using the Schwarz inequality we obtain

$$\int_M \|\Phi_n\|^2 \leq \left(\text{Area}(M) \cdot \int_M \|\Phi_n\|^4 \right)^{\frac{1}{2}}.$$

Thus integrating (4.10) we have

$$\|[\Phi_n, \Phi_n^*]\|_{L^2}^2 + 4 \int_M |\det \Phi_n|^2 \geq \frac{1}{\text{Area}(M)} \cdot \|\Phi_n\|_{L^2}^4. \quad (4.11)$$

However, each Φ_n is on the same orbit under the group of complex gauge transformations acting by conjugation, so

$$\det \Phi_n = \det \Phi_1 \in H^0(M; K^2).$$

Thus from (4.9) and (4.11) we have

$$\|\Phi_n\|_{L^2}^4 \leq K_1 + K_2 \|\Phi_n\|_{L^2}^2 \quad \text{for some } K_1, K_2 > 0,$$

and so

$$\|\Phi_n\|_{L^2} \leq K. \quad (4.12)$$

Equation (4.9) now gives a uniform L^2 bound on $[\Phi_n, \Phi_n^*]$ and (4.4) then gives an L^2 bound on $F(A_n)$.

From Uhlenbeck's theorem, after applying unitary gauge transformations, we may assume that A_n converges weakly in L_1^2 to a connection A .

We now require L_1^2 bounds on Φ_n . The inequality (4.10) gives an L^4 bound. However, Φ_n satisfies the elliptic equation

$$d_{A_n}'' \Phi_n = 0$$

which may be written in terms of a fixed C^∞ d'' -operator as

$$d'' \Phi_n + [\alpha_n, \Phi_n] = 0. \quad (4.13)$$

Since A_n is bounded in L_1^2 and $L_1^2 \subset L^4$ is compact, we have an L^4 bound on α_n and Φ_n and hence an L^2 bound on $[\alpha_n, \Phi_n]$. However, for the elliptic operator d'' we have the estimate

$$\|\Phi_n\|_{L_1^2} \leq C(\|d'' \Phi_n\|_{L^2} + \|\Phi_n\|_{L^2})$$

and so from (4.12) and (4.13) we obtain an L_1^2 bound on Φ_n . By the weak compactness of L_1^2 , this has a weakly convergent subsequence tending to Φ . In order to complete the proof we must show that (A, Φ) lies on the same orbit as

(A_n, Φ_n) . To this end define the operator

$$d''_{A_n A_1}: \Omega^0(M; V^* \otimes V) \rightarrow \Omega^{0,1}(M; V^* \otimes V)$$

by using the connection A_n on the V^* factor and A_1 on the V factor. Hence

$$d''_{AA_1} = d''_{A_n A_1} + \beta_n$$

where $\beta_n \rightarrow 0$ weakly in L^2_1 . The elliptic estimate for d''_{AA_1} is

$$\|\psi\|_{L^2_1} \leq C(\|d''_{AA_1}\psi\|_{L^2} + \|\psi\|_{L^2}).$$

Suppose now that $d''_{A_n A_1}\psi_n = 0$ and $\|\psi_n\|_{L^2} = 1$. Then

$$\begin{aligned} \|\psi_n\|_{L^2_1} &\leq C(\|[\beta_n, \psi_n]\|_{L^2} + 1) \\ &\leq K_1 \|\beta_n\|_{L^4} \|\psi_n\|_{L^4} + K_2. \end{aligned}$$

Since $L^2_1 \subset L^4$ is compact, $\|\beta_n\|_{L^4} \rightarrow 0$ and so we obtain an L^2_1 bound on ψ_n which therefore has a weakly convergent subsequence. Since $\|\psi_n\|_{L^2} = 1$ and $L^2_1 \subset L^2$ is compact, this weak limit ψ is non-zero.

In our case, take ψ_n to be the complex automorphism which maps (A_n, Φ_n) to (A_1, Φ_1) , normalized in L^2 . This by definition satisfies

$$d''_{A_n A_1}\psi_n = 0,$$

and furthermore satisfies the algebraic identity

$$\Phi_1\psi_n - \psi_n\Phi_n = 0. \quad (4.14)$$

Since ψ_n and Φ_n converge weakly in L^2_1 and $L^2_1 \subset L^4$ is compact,

$$0 = \|\Phi_1\psi_n - \psi_n\Phi_n\|_{L^2} \rightarrow \|\Phi_1\psi - \psi\Phi\|_{L^2}$$

and so, in the limit,

$$\Phi_1\psi - \psi\Phi = 0 \quad \text{and} \quad d''_{AA_1}\psi = 0. \quad (4.15)$$

If ψ were an isomorphism, then it would be the required complex gauge transformation to place (A, Φ) on the same orbit as (A_1, Φ_1) . If ψ is not an isomorphism, then ψ maps V into a subbundle L of V , holomorphic with respect to d''_{A_1} . Moreover by (4.15), L is invariant by Φ_1 . Since (A_1, Φ_1) is stable, then by Proposition (3.4) this is impossible for Φ_1 in a Zariski open set. Thus, for a generic Φ_1 , (A, Φ) is on the same orbit and is a minimum for the functional f and consequently gives a solution to the equation

$$F(A) + [\Phi, \Phi^*] = 0. \quad (4.16)$$

Note now some consequences of the inequalities established above when the equation (4.16) is satisfied. From (4.6),

$$\|[\Phi, \Phi^*]\|_{L^2}^2 \leq 4c \|\Phi\|_{L^2}^2, \quad (4.17)$$

and from (4.11) and (4.17),

$$\|\Phi\|_{L^2}^4 \leq c_1 + c_2 \int_M |\det \Phi|^2 \quad (4.18)$$

for positive constants c_1 and c_2 .

Fix a complex structure A_0 on V , and consider $\Phi \in H^0(M; \text{End}_0 V \otimes K)$. We

have shown that for a Zariski open (and hence dense) set

$$U \subseteq H^0(M ; \text{End}_0 V \otimes K)$$

there is a solution to the self-duality equations equivalent under an automorphism to (A_0, Φ) . Now suppose $\Phi_n \in U$ and, in the finite-dimensional vector space $H^0(M ; \text{End}_0 V \otimes K)$, suppose Φ_n tends to Φ_0 . Then in particular,

$$\det \Phi_n \rightarrow \det \Phi_0 \in H^0(M ; K^2)$$

and so, from (4.17) and (4.18), $[\Phi_n, \Phi_n^*] = -F(A_n)$ is bounded in L^2 . Therefore, by the arguments above, we may assume after applying unitary gauge transformations, that (A_n, Φ_n) tends weakly to (A, Φ) in L^2 . However, (A_n, Φ_n) satisfies $F(A_n) + [\Phi_n, \Phi_n^*] = 0$, which is preserved under weak limits, so (A, Φ) also satisfies these equations. We need to show that if (A_0, Φ_0) is stable, then (A, Φ) is in the same orbit under complex automorphisms.

Now the complex structures A_n are all equivalent to A_0 so we have complex gauge transformations ψ_n in L^2_2 such that

$$d''_{A_n A_0} \psi_n = 0$$

and

$$\psi_n \Phi_n \psi_n^{-1} \rightarrow \Phi_0 \tag{4.19}$$

by assumption. This convergence is in the finite-dimensional vector space $H^0(M ; \text{End}_0 V \otimes K)$, and so holds for any norm.

Repeating the argument above, we see that if ψ_n is normalized in L^2 , then $\psi_n \rightarrow \psi$ weakly in L^2_1 where

$$d''_{A A_1} \psi = 0. \tag{4.20}$$

Furthermore, since

$$\|\psi_n \Phi_n - \Phi_0 \psi_n\|_{L^2} \leq C \|\psi_n\|_{L^2_1} \|\psi_n \Phi_n \psi_n^{-1} - \Phi_0\|_{L^2_1}$$

and ψ_n is bounded in L^2_1 and furthermore (4.19) holds, we deduce, using the compact inclusion $L^2_1 \subset L^4$, that

$$\psi \Phi - \Phi_0 \psi = 0. \tag{4.21}$$

We wish to show that ψ is an isomorphism, but this follows from Proposition (3.15) since (A_0, Φ_0) is stable by assumption and (A, Φ) is stable from Theorem (2.1), since it satisfies the self-duality equation. Thus ψ must be an isomorphism and (A, Φ) is equivalent to (A_0, Φ_0) .

The uniqueness in (4.3) follows from Theorem (2.7).

COROLLARY (4.22) (Narasimhan and Seshadri). *Every stable rank-2 bundle V over a compact Riemann surface M of genus $g > 1$ is associated to a flat $\text{SO}(3)$ connection, unique up to gauge transformations.*

Proof. Take $\Phi = 0$; then (V, Φ) is stable only if V is stable. The equation $F + [\Phi, \Phi^*] = 0$ yields $F = 0$.

Note that although the above proof follows Donaldson's half-way, the use of the auxiliary field Φ in the generic case leads to a quite short conclusion. The extra tool to pass from the generic to the special case is a further application of Uhlenbeck's theorem.

COROLLARY (4.23) (Riemann, Poincaré, and Koebe). *Every compact Riemann surface of genus $g > 1$ admits a metric of constant negative curvature.*

Proof. Take $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, and $\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as in (3.2)(ii). This is a stable pair and Theorem (4.3) provides a solution to $F + [\Phi, \Phi^*] = 0$, unique modulo unitary gauge transformations.

Now if (A, Φ) solves the self-duality equation, so does $(A, e^{i\theta}\Phi)$ for a constant θ . However, in this instance, the automorphism of the complex structure of V ,

$$\psi = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix},$$

gives

$$\psi^{-1}\Phi\psi = e^{i\theta}\Phi$$

and so (A, Φ) and $(A, e^{i\theta}\Phi)$ are in the same orbit under complex gauge transformations. By uniqueness, they must be gauge-equivalent solutions. This implies that the connection A has a 1-parameter group of automorphisms and must be reducible to a $U(1)$ connection. The relationship with metrics of constant negative curvature is given in Example (1.5).

5. The moduli space

We have proved in (4.3) an existence theorem for solutions of the self-duality equations on a Riemann surface. We shall now consider the space of all solutions on a fixed principal bundle P , modulo the group of gauge transformations. This is a setting which is by now familiar in the context of gauge theories on 4-manifolds [4, 7, 30] or stable bundles on Riemann surfaces [5]. The basic idea is to linearize the equations and consider the elliptic complex which arises from this. An application of the Atiyah–Singer index theorem and a vanishing theorem will yield the dimension of the linearization. Slice theorems and elliptic estimates then show that the moduli space is a manifold of the calculated dimension.

We begin with the part which is specific to this problem: the linearization of the self-duality equation (1.3)

$$\left. \begin{aligned} F(A) &= -[\Phi, \Phi^*], \\ d''_A \Phi &= 0. \end{aligned} \right\}$$

If $(\dot{A}, \dot{\Phi}) \in \Omega^1(M; \text{ad } P) \oplus \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$ denotes an infinitesimal variation, then the equations are satisfied to first order if

$$\left. \begin{aligned} d_A \dot{A} &= -[\dot{\Phi}, \Phi^*] - [\Phi, \dot{\Phi}^*], \\ d''_A \dot{\Phi} + [\dot{A}^{0,1}, \Phi] &= 0. \end{aligned} \right\} \tag{5.1}$$

The infinitesimal variation arises from an infinitesimal gauge transformation $\dot{\psi} \in \Omega^0(M; \text{ad } P)$ if

$$\left. \begin{aligned} \dot{A} &= d_A \dot{\psi}, \\ \dot{\Phi} &= [\Phi, \dot{\psi}]. \end{aligned} \right\} \tag{5.2}$$

We obtain this way a complex

$$\begin{aligned} \Omega^0(M ; \text{ad } P) &\xrightarrow{d_1} \Omega^1(M ; \text{ad } P) \oplus \Omega^{1,0}(M ; \text{ad } P \otimes \mathbb{C}) \\ &\xrightarrow{d_2} \Omega^2(M ; \text{ad } P) \oplus \Omega^2(M ; \text{ad } P \otimes \mathbb{C}), \end{aligned} \quad (5.3)$$

where

$$\begin{aligned} d_1 \dot{\psi} &= (d_A \dot{\psi}, [\Phi, \dot{\psi}]), \\ d_2(\dot{A}, \dot{\Phi}) &= (d_A \dot{A} + [\dot{\Phi}, \Phi^*] + [\Phi, \dot{\Phi}^*], d_A'' \dot{\Phi} + [\dot{A}^{0,1}, \Phi]). \end{aligned}$$

Considering the highest-order part of the differential operators d_1 and d_2 , we see that the complex is elliptic and has the same index as the direct sum of the two complexes (for $G = \text{SO}(3)$),

$$\begin{aligned} \Omega^0(M ; \mathbb{R}^3) &\xrightarrow{d} \Omega^1(M ; \mathbb{R}^3) \xrightarrow{d} \Omega^2(M ; \mathbb{R}^3), \\ 0 &\longrightarrow \Omega^{1,0}(M ; \mathbb{C}^3) \xrightarrow{d''} \Omega^{1,1}(M ; \mathbb{C}^3), \end{aligned}$$

which is (considering real dimensions) given by

$$\text{index} = 3(2 - 2g) - 6(g - 1) = 12(1 - g). \quad (5.4)$$

The vector space of infinitesimal deformations of the equation modulo infinitesimal gauge transformations is the first cohomology group of the elliptic complex (5.3), and (5.4) says that

$$\dim H^0 - \dim H^1 + \dim H^2 = 12(1 - g).$$

Now $H^0 = \ker d_1$ consists of the space of $\psi \in \Omega^0(M ; \text{ad } P)$ which are covariant constant with respect to the connection A , and commute with Φ . If $H^0 \neq 0$, then the solution to the self-duality equations must be reducible. To deal with H^2 , we must show that $\ker d_2^* = 0$, and this can be done by considering the single operator $d_2^* + d_1$. In fact if we pair the operator d_2^* acting on the real space

$$\Omega^2(M ; \text{ad } P) \cong \Omega^0(M ; \text{ad } P)$$

with d_1 into a complex operator acting on $\Omega^0(M ; \text{ad } P \otimes \mathbb{C})$, and identify $\Omega^1(M ; \text{ad } P)$ with the complex space $\Omega^{0,1}(M ; \text{ad } P \otimes \mathbb{C})$, then we define

$$\begin{aligned} d_2^* + d_1: \Omega^0(M ; \text{ad } P \otimes \mathbb{C}) \oplus \Omega^0(M ; \text{ad } P \otimes \mathbb{C}) \\ \rightarrow \Omega^{0,1}(M ; \text{ad } P \otimes \mathbb{C}) \oplus \Omega^{1,0}(M ; \text{ad } P \otimes \mathbb{C}). \end{aligned}$$

Calculating the adjoint of d_2 , we find that $(d_2^* + d_1)(\psi_1, \psi_2) = 0$ if and only if

$$\left. \begin{aligned} d_A'' \psi_1 + [\Phi^*, \psi_2] &= 0, \\ d_A' \psi_2 + [\Phi, \psi_1] &= 0. \end{aligned} \right\} \quad (5.5)$$

Now from (5.5),

$$\begin{aligned} d_A' d_A'' \psi_1 &= [\Phi^*, d_A' \psi_2] \quad (\text{since } d_A' \Phi^* = 0) \\ &= -[\Phi^*, [\Phi, \psi_1]] \quad (\text{from (5.5)}). \end{aligned}$$

Thus since

$$d' \langle d_A'' \psi_1, \psi_1 \rangle = \langle d_A' d_A'' \psi_1, \psi_1 \rangle - \langle d_A'' \psi_1, d_A'' \psi_1 \rangle,$$

we have

$$d' \langle d''_A \psi_1, \psi_1 \rangle = - \langle [\Phi^*, [\Phi, \psi_1]], \psi_1 \rangle - \langle d''_A \psi_1, d''_A \psi_1 \rangle$$

and, on integrating,

$$0 = \int_M \|[\Phi, \psi_1]\|^2 + \int_M \|d''_A \psi_1\|^2,$$

so that $d''_A \psi_1 = 0$ and $[\Phi, \psi_1] = 0$.

Now apply the vanishing method of Theorem (2.7) to the holomorphic section ψ_1 of $\text{ad } P \otimes \mathbb{C}$ and we obtain

$$d'_A \psi_1 = 0 \quad \text{and} \quad [\Phi^*, \psi_1] = 0.$$

If the solution is irreducible, then the covariant constant section ψ_1 , commuting with Φ , must vanish.

Repeating the argument with ψ_2 , we see similarly that ψ_2 must vanish identically.

We have thus proved that, at an irreducible solution to the self-duality equations, the 0th and 2nd cohomology groups of the complex (5.3) vanish, and hence in particular, from (5.4),

$$\dim H^1 = 12(g - 1). \tag{5.6}$$

Now we pass to the general setting of moduli spaces. We consider, as in § 4, the space

$$\mathcal{A}(M; P) \times \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C}) = \mathcal{A} \times \Omega,$$

the product of the space of L^2_1 connections on P with L^2_1 sections of $\text{ad } P \otimes_{\mathbb{C}} K$, and the action of the group \mathcal{G} of L^2_2 $\text{SO}(3)$ -gauge transformations on it. This is a Banach manifold with the smooth action of a Banach Lie group on it [5, 30].

A *slice* for the action may be found at a regular point (A, Φ) , that is, one for which the group \mathcal{G} of gauge transformations has isotropy group the identity. This implies that there are no non-zero solutions to $d_1 \psi = 0$ where $d_1 \psi = (d_A \psi, [\Phi, \psi])$.

The slice is defined as the kernel of d_1^* , and the slices provide coordinate patches for the quotient

$$(\mathcal{A} \times \Omega)_0 / \mathcal{G} = \mathcal{B}$$

of the open set of regular points by the group of gauge transformations. The quotient space \mathcal{B} , with the quotient topology, has the structure of a Banach manifold. Details for this theorem in dimension 4, which do not differ significantly for the case of dimension 2, were worked out by Parker [30], based on analogous questions for connections alone [27, 7].

The self-duality equations,

$$\left. \begin{aligned} F(A) + [\Phi, \Phi^*] &= 0, \\ d''_A \Phi &= 0, \end{aligned} \right\}$$

are in this context the zero set of a smooth section of a vector bundle over \mathcal{B} . In a local coordinate system defined by a slice, this section is defined by the function

$$\begin{aligned} f: \ker d_{1A_0}^* &\rightarrow \Omega^2(M; \text{ad } P) \oplus \Omega^2(M; \text{ad } P \otimes \mathbb{C}), \\ f(A, \Phi) &= (F(A) + [\Phi, \Phi^*], d''_A \Phi). \end{aligned}$$

At $A = A_0$, the derivative of f is the linearized operator d_2 of the elliptic complex (5.3) restricted to $\ker d_1^*$. We showed that $\ker(d_2^* + d_1) = 0$ and so using elliptic regularity, we see that df_{A_0} is surjective as a Banach space transformation. Using the Banach space implicit function theorem, we deduce that the zero set of f is a smooth submanifold of the slice of dimension $12(g - 1)$.

We have thus seen that the points on which the group \mathcal{G} of gauge transformations acts with trivial isotropy subgroup form a smooth manifold. We now have to distinguish between $U(2)$ and $SO(3)$ gauge transformations in order to be able to consider the global structure of the moduli space. From Proposition (3.15) the group \mathcal{G}_0 of $U(2)$ gauge transformations with unit determinant acts on stable pairs (V, Φ) with isotropy ± 1 . Hence taking $\mathcal{G} = \mathcal{G}_0/\pm 1$ acting on pairs (A, Φ) of solutions to the self-duality equations, where A is a connection on the vector bundle V , gives a smooth moduli space. On the other hand, if we use $SO(3)$ gauge transformations, we will necessarily encounter non-trivial isotropy subgroups even restricting to stable pairs, as follows.

The basic example of this is to consider the moduli space \mathcal{N} of stable rank-2 bundles with fixed determinant and odd degree. This (see [5]) is a compact smooth projective variety with $c_1 > 0$. By Kodaira's vanishing theorem $H^p(\mathcal{N}, \mathcal{O}) = 0$ for $p > 0$, whence the Todd genus is 1. If G is a finite group of biholomorphic transformations acting freely on \mathcal{N} , then $c_1(\mathcal{N}/G) > 0$ and so the Todd genus of \mathcal{N}/G is similarly 1. As pointed out by Kobayashi, this contradicts the multiplicativity of Todd genus and so G must have fixed points.

In the particular case of the moduli space \mathcal{N} , the action of tensoring a stable bundle V by a line bundle L of order 2 gives an action of \mathbb{Z}_2^{2g} . Hence for each such line bundle there must be a fixed point, i.e. a stable bundle V such that $V \cong V \otimes L$. The corresponding projective bundles are identical, so using the theorem of Narasimhan and Seshadri, we have a flat $SO(3)$ connection with a non-trivial automorphism: the connection is reduced to $O(2)$ and so the group of $SO(3)$ gauge transformations has non-trivial isotropy.

For this reason we shall find it more convenient to use the moduli space corresponding to connections on the vector bundle V rather than the associated principal $SO(3)$ bundle.

In fact, if the $SO(3)$ connection reduces to $O(2)$, the bundle $\text{End}_0 V$ decomposes as

$$\text{End}_0 V = L \oplus U,$$

where L is a flat line bundle of order 2, and U a real rank-2 bundle, and then the only Higgs fields invariant by the non-trivial automorphism are obtained by taking a holomorphic section of LK on M . We then always have $[\Phi, \Phi^*] = 0$ and hence $F = 0$, so that the $O(2)$ connection is flat also.

We consequently obtain the following theorem.

THEOREM (5.7). *Let V be a rank-2 vector bundle of odd degree over a compact Riemann surface M of genus $g > 1$, and let \mathcal{M} be the moduli space of solutions to the self-duality equations on V , with fixed induced connection on $\Lambda^2 V$. Then \mathcal{M} is a smooth manifold of dimension $12(g - 1)$.*

Proof. If V has odd degree, there are no solutions reducible to $U(1)$; hence \mathcal{G} acts freely.

The existence theorem (4.3) yields the corresponding algebro-geometric result:

THEOREM (5.8). *Let M be a compact Riemann surface of genus $g > 1$. The moduli space of all stable pairs (V, Φ) , where V is a rank-2 holomorphic vector bundle of fixed determinant and odd degree, and Φ is a trace-free holomorphic section of $\text{End } V \otimes K$, is a smooth manifold of real dimension $12(g - 1)$.*

Clearly we expect the moduli space to be a complex manifold of dimension $6(g - 1)$. Actually, much more is true as we shall see on considering the natural metric on \mathcal{M} in § 6.

6. The Riemannian structure of the moduli space

From its very definition, the tangent space at a point m of \mathcal{M} is naturally isomorphic to the vector subspace W of $\Omega^1(M; \text{ad } P) \oplus \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$ consisting of all $(\dot{A}, \dot{\Phi})$ such that

$$\left. \begin{aligned} d_A \dot{A} + [\dot{\Phi}, \Phi^*] + [\Phi, \dot{\Phi}^*] &= 0, \\ d_A'' \dot{\Phi} + [\dot{A}^{0,1}, \Phi] &= 0, \\ d_A^* \dot{A} + \text{Re}[\Phi^*, \dot{\Phi}] &= 0, \end{aligned} \right\} \quad (6.1)$$

for a representative (A, Φ) of m . The last equation is $d_1^*(\dot{A}, \dot{\Phi}) = 0$, which means that the space is orthogonal in L^2 to the orbits under the group of gauge transformations.

Now the space $\mathcal{A} \times \Omega = \mathcal{A}(M; \text{ad } P) \times \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$ has, as we saw in § 4, a natural Kähler metric

$$g((\psi, \Phi), (\psi, \Phi)) = 2i \int_M \text{Tr}(\psi^* \psi + \Phi \Phi^*), \quad (6.2)$$

where we identify $\Omega^1(M; \text{ad } P)$ with the complex space $\Omega^{0,1}(M; \text{ad } P \otimes \mathbb{C})$. Thus W has a natural induced inner product which, since it is invariant under the action of the group of unitary gauge transformations, induces an inner product on the tangent space at m . Thus the moduli space \mathcal{M} has a natural metric.

THEOREM (6.1). *Let M be a compact Riemann surface of genus $g > 1$ and \mathcal{M} the moduli space of solutions to the self-duality equations on a rank-2 vector bundle V of odd degree. Then the natural metric on \mathcal{M} is complete.*

Proof. Assume \mathcal{M} is not complete. Then there exists a geodesic $\gamma(s)$ of finite length, i.e. if γ is parametrized by arc length, $\{s \in \mathbb{R} \mid \gamma(s) \in \mathcal{M}\}$ is bounded above, with supremum \bar{s} , say. Now let $\tilde{\mathcal{M}} \subset \mathcal{A} \times \Omega$ be the Banach submanifold consisting of all solutions to the self-duality equations. This is a principal \mathcal{G} -bundle over \mathcal{M} . The curve $\gamma(s)$ may be lifted to a horizontal curve $\tilde{\gamma}(s)$ in $\tilde{\mathcal{M}}$. This follows from the existence theorem for differential equations in Banach spaces (see [22]).

By definition of the metric, the length of $\tilde{\gamma}$ from $s = s_0$ to $s = s_n$ is the same as the length of γ , that is, $|s_n - s_0|$, which is bounded above. This length is greater than or equal to the straight line distance in the metric (6.2).

Hence we have a bound

$$\|A(s_n) - A(s_0)\|_{L^2}^2 + \|\Phi(s_n) - \Phi(s_0)\|_{L^2}^2 \leq M$$

as $s_n \rightarrow \bar{s}$.

In particular,

$$\|\Phi(s_n)\|_{L^2} \leq M^{\frac{1}{2}} + \|\Phi(s_0)\|_{L^2}. \tag{6.4}$$

However, from (4.17) this gives an L^2 bound on $[\Phi, \Phi^*]$ and hence $F(A(s_n))$.

By Uhlenbeck's theorem, after applying gauge transformations (which do not alter $\gamma(s_n) \in \mathcal{M}$), we may take $(A(s_n), \Phi(s_n))$ to converge weakly in L^2_1 to a solution (A, Φ) of the self-duality equations.

If $w_2 \neq 0$, this can never be reducible to $U(1)$ and therefore corresponds to a point m in the manifold \mathcal{M} .

Considering the continuous projection onto the slice at (A, Φ) , which is a finite-dimensional vector space, we see that the weakly convergent sequence becomes a sequence converging to (A, Φ) .

Hence $\gamma(s_n) \rightarrow m$ in the topology of \mathcal{M} . Standard differential geometric arguments (see [14]) show this to be a contradiction.

The natural metric on \mathcal{M} is a very special one—it is a *hyperkähler metric*. Recall that a hyperkähler metric on a $4n$ -dimensional manifold is a Riemannian metric which is Kählerian with respect to three complex structures I, J, K which satisfy the algebraic identities of the quaternions:

$$\left. \begin{aligned} I^2 = J^2 = K^2 = -1, \\ IJ = -JI = K, \\ JK = -KJ = I, \\ KI = -IK = J. \end{aligned} \right\}$$

Corresponding to each complex structure is a Kähler form:

$$\omega_1(X, Y) = g(IX, Y), \quad \omega_2(X, Y) = g(JX, Y), \quad \omega_3(X, Y) = g(KX, Y),$$

and furthermore this set of symplectic forms determines the metric uniquely. It is the symplectic aspect of the self-duality equations which leads to the fact that the natural metric is hyperkählerian.

We have already seen this in the approach to the existence theorem (4.3). There, the equation

$$F + [\Phi, \Phi^*] = 0$$

was interpreted as the zero set of the moment map corresponding to the action of the gauge group \mathcal{G} on the Kähler manifold $\mathcal{A} \times \Omega$ with Kähler metric (6.2).

We shall now interpret the other half of the self-duality equations

$$d''_A \Phi = 0$$

in terms of moment maps.

To see this, note that the tangent space to $\mathcal{A} \times \Omega$ at (A, Φ) is naturally

$$\Omega^{0,1}(M; \text{ad } P \otimes \mathbb{C}) \oplus \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C}).$$

We define a complex symplectic form ω on this space by

$$\omega((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \int_M \text{Tr}(\Phi_2 \psi_1 - \Phi_1 \psi_2). \quad (6.5)$$

Since this has constant coefficients, it is clearly a closed form on the affine space $\mathcal{A} \times \Omega$. Let $\psi \in \Omega^0(M; \text{ad } P)$ be an infinitesimal gauge transformation; then it defines the vector field $X = (\Psi_1, \Phi_1)$ on $\mathcal{A} \times \Omega$ by

$$\Psi_1 = d''_A \psi, \quad \Phi_1 = [\Phi, \psi].$$

Consequently, by (6.5),

$$\begin{aligned} i(X)\omega(\dot{A}^{0,1}, \dot{\Phi}) &= \int_M \text{Tr}(-[\Phi, \psi]\dot{A}^{0,1} + \dot{\Phi}d''_A \psi) \\ &= \int_M \text{Tr}(\psi[\dot{A}^{0,1}, \Phi] + d''_A \dot{\Phi}) \\ &= df(\dot{A}^{0,1}, \dot{\Phi}), \end{aligned}$$

where

$$f = \int_M \text{Tr}(d''_A \Phi \psi).$$

Hence $d''_A \Phi = 0$ is the zero set of the moment maps of the two symplectic forms given by the real and imaginary parts of ω , with respect to the natural action of the group \mathcal{G} of gauge transformations. Call these forms ω_2 and ω_3 and the Kähler form of the metric (6.2) ω_1 . Then $\omega_1, \omega_2, \omega_3$ are the Kähler forms for a flat hyperkähler metric on $\mathcal{A} \times \Omega$. This is easy to see because they all have constant coefficients.

Furthermore, if $\mu_1, \mu_2,$ and μ_3 denote the corresponding moment maps with respect to \mathcal{G} , then the self-duality equations are given by

$$\mu_i(A, \Phi) = 0, \quad \text{where } 1 \leq i \leq 3,$$

and the moduli space \mathcal{M} is the quotient

$$\mathcal{M} = \bigcap_{i=1}^3 \mu_i^{-1}(0)/\mathcal{G}. \quad (6.6)$$

It is a theorem, in finite dimensions [17, 18], that the natural metric on the quotient in this sense (a generalization of the Marsden–Weinstein quotient in symplectic geometry) of a hyperkähler manifold is again hyperkähler. We shall adapt the proof slightly to deal with the present situation:

THEOREM (6.7). *Let M be a compact Riemann surface of genus $g > 1$ and \mathcal{M} the moduli space of irreducible solutions to the $\text{SO}(3)$ self-duality equations. Then the natural metric on the $12(g-1)$ -dimensional manifold \mathcal{M} is hyperkählerian.*

Proof. Let X be a tangent vector in $\mathcal{A} \times \Omega$, tangent to the submanifold $\tilde{\mathcal{M}}$ of solutions to the self-duality equations. Then, in terms of the moment maps $\mu_1, \mu_2,$ and $\mu_3,$

$$d\mu_i(X) = 0 \quad \text{for } 1 \leq i \leq 3.$$

Thus by the definition of the moment map,

$$g(IX, Y) = g(JX, Y) = g(KX, Y) = 0$$

for all vectors Y tangent to the orbit of \mathcal{G} , where I, J, K are the constant complex structures on $\mathcal{A} \times \Omega$. Thus IX, JX , and KX are orthogonal to the orbit. Clearly, then, the horizontal space W is preserved by I, J , and K .

Since I, J, K are also preserved by \mathcal{G} , we see that the tangent space at a point $m \in \mathcal{M}$ admits an action of the quaternions, compatible with the metric. Call such a structure an *almost hyperkähler metric*.

We need to prove the integrability of I, J, K and then the closure of $\omega_1, \omega_2, \omega_3$, but fortunately these can be treated together in the following lemma.

LEMMA (6.8). *Let g be an almost hyperkähler metric, with 2-forms $\omega_1, \omega_2, \omega_3$ corresponding to almost complex structures I, J, K . Then g is hyperkähler if each ω_i is closed.*

Proof. First note that

$$\omega_2(X, Y) = g(JX, Y) = g(KIX, Y) = \omega_3(IX, Y)$$

and hence

$$i(X)\omega_2 = i(IX)\omega_3. \quad (6.9)$$

If X is a complex vector field, then it follows that $IX = iX$ if and only if $i(X)\omega_2 = i(i(X)\omega_3)$. We wish to show first that I is integrable, so suppose $IX = iX$ and $IY = iY$. Then

$$\begin{aligned} i([X, Y])\omega_2 &= \mathcal{L}_X(i(Y)\omega_2) - i(Y)\mathcal{L}_X\omega_2 \\ &= \mathcal{L}_X(i(i(Y)\omega_3)) - i(Y)\mathcal{L}_X\omega_2 \quad (\text{from (6.9)}) \\ &= \mathcal{L}_X(i(i(Y)\omega_3)) - i(Y)d(i(X)\omega_2) \quad (\text{since } d\omega_2 = 0) \\ &= \mathcal{L}_X(i(i(Y)\omega_3)) - i(Y)d(i(i(X)\omega_3)) \quad (\text{from (6.9)}) \\ &= \mathcal{L}_X(i(i(Y)\omega_3)) - i(i(Y)\mathcal{L}_X\omega_3) \quad (\text{since } d\omega_3 = 0) \\ &= i([X, Y]\omega_3). \end{aligned}$$

Hence, from (6.9),

$$I[X, Y] = i[X, Y],$$

so by the Newlander–Nirenberg theorem I is integrable. Since $d\omega_1 = 0$, g is Kähler with respect to I .

Repeating with J and K , we obtain the lemma.

Returning to the theorem, note that if $d\mu_i(X) = 0$, then by the definition of the moment map, $\omega_i(X, Y) = 0$ for all Y tangent to the \mathcal{G} -orbit. Hence if $\bar{\omega}_i$ denote the 2-forms obtained from I, J, K on \mathcal{M} , then

$$p^*\bar{\omega}_i = \omega_i|_{\tilde{\mathcal{M}}}$$

where $p: \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ is the projection.

Since $d\omega_i = 0$, $p^*d\bar{\omega}_i = 0$ and thus, since p is a fibration, $d\bar{\omega}_i = 0$.

Therefore $\bar{\omega}_i$ are all closed, and from (6.8) the metric on \mathcal{M} is hyperkählerian.

There is one more piece of Riemannian information to note. If (A, Φ) is a solution to the self-duality equations

$$\left. \begin{aligned} F + [\Phi, \Phi^*] &= 0, \\ d_A''\Phi &= 0, \end{aligned} \right\}$$

then clearly so is $(A, e^{i\theta}\Phi)$ for a constant θ . Moreover, this action of the circle preserves the metric g of (6.2) and thus acts as isometries on \mathcal{M} . It preserves the standard complex structure I on $\mathcal{A} \times \Omega$, but not the other two complex structures J and K since it does not leave the complex symplectic form (6.5) invariant.

Since the action preserves the equations, and commutes with the action of the group of gauge transformations, it descends to a circle action on \mathcal{M} , acting by isometries of the natural metric. Since it preserves the Kähler form $\bar{\omega}_1$, it has a moment map μ , which we may calculate on $\mathcal{A} \times \Omega$.

The vector field generated by the action is

$$(\dot{A}, \dot{\Phi}) = (0, i\Phi).$$

Hence

$$i(X)\omega_1(Y) = g(IX, Y) = g(-\Phi, Y) = -\frac{1}{2}dg(\Phi, \Phi)(Y).$$

Thus the moment map for the action is $-\frac{1}{2}\|\Phi\|_{L^2}^2$. We shall use this function next to analyse the topology of the moduli space.

7. The topology of the moduli space

We consider the manifold \mathcal{M} , the moduli space of solutions to the self-duality equations on a rank-2 vector bundle V of odd degree. From the complex structures I, J, K put on \mathcal{M} the complex structure I invariant by the action of the circle $(A, \Phi) \rightarrow (A, e^{i\theta}\Phi)$. This is the natural complex structure which \mathcal{M} inherits under its interpretation through Theorem (4.3) as the moduli space of stable pairs (V, Φ) , where V is a rank-2 vector bundle of odd degree and fixed determinant. We shall investigate the algebraic topology of \mathcal{M} using the Morse function

$$\mu(A, \Phi) = 2i \int_M \text{Tr}(\Phi\Phi^*) = \|\Phi\|_{L^2}^2.$$

The method is due to Frankel [10] (see also [5]), and uses the fact that, since by (6.10),

$$d\mu = -2i(X)\omega_1,$$

the critical points of μ are the fixed points of the circle action generated by the vector field X .

PROPOSITION (7.1). *The function $\mu = \|\Phi\|_{L^2}^2$ on \mathcal{M} has the following properties.*

- (i) μ is proper.
- (ii) μ has critical values 0 and $(d - \frac{1}{2})\pi$ where d is a positive integer less than or equal to $g - 1$.
- (iii) $\mu^{-1}(0)$ is a non-degenerate critical manifold of index 0, and is diffeomorphic to the moduli space of stable rank-2 bundles of odd degree and fixed determinant over M .
- (iv) $\mu^{-1}((d - \frac{1}{2})\pi)$ is a non-degenerate critical manifold of index $2(g + 2d - 2)$,

and is diffeomorphic to a 2^{2g} -fold covering of the $(2g - 2d - 1)$ -fold symmetric product $S^{2g-2d-1}M$ of the Riemann surface. The covering is the pull-back of the covering $\text{Jac}(M) \rightarrow \text{Jac}(M)$ given by $x \mapsto 2x$ under the natural map $S^{2g-2d-1}M \rightarrow \text{Jac}(M)$ which associates to a $(2g - 2d - 1)$ -tuple of points of M its divisor class.

Proof. (i) From (4.17), if $\mu \leq k$, we have an L^2 bound on the curvature F of the solutions to the self-duality equations in $\mu^{-1}[0, k]$.

Using Uhlenbeck's theorem, note that any infinite sequence in $\mu^{-1}[0, k]$ has a convergent subsequence; hence $\mu^{-1}[0, k]$ is compact.

(ii) As remarked above, the critical points of μ are the fixed points of the circle action

$$(A, \Phi) \rightarrow (A, e^{i\theta}\Phi)$$

induced on \mathcal{M} .

Clearly $\Phi = 0$ is a fixed point and this occurs if and only if $\mu = \|\Phi\|_{L^2}^2 = 0$. The self-duality equations then become

$$F = 0,$$

and thus, by the theorem of Narasimhan and Seshadri, $\mu^{-1}(0)$ is the moduli space of stable holomorphic vector bundles of rank-2 and odd degree.

There are no more fixed points on the space $\mathcal{A} \times \Omega$ but we must remember that we are considering the circle action on the quotient space by the group of gauge transformations. Thus (A, Φ) represents a fixed point if there are gauge transformations $g(\theta)$ such that

$$\left. \begin{aligned} g(\theta)^{-1}\Phi g(\theta) &= e^{i\theta}\Phi, \\ g(\theta)^{-1}d_A g(\theta) &= d_A. \end{aligned} \right\} \tag{7.2}$$

If $\Phi \neq 0$, then $g(\theta)$ is not the identity for $\theta \neq 2k\pi$, and so the second equation implies that A is reducible to a $U(1)$ connection. This means that the associated vector bundle V is decomposable:

$$V = L \oplus L^*\Lambda^2V.$$

Since $g(\theta)$ is diagonal with respect to this decomposition, from (7.2), Φ must be lower triangular with respect to one ordering of the decomposition. Let

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}, \quad \text{where } \phi \in \Omega^0(M; L^{-2}K \otimes \Lambda^2V).$$

By the self-duality equation, ϕ is holomorphic and $F + [\Phi, \Phi^*] = 0$. Thus, in $\Omega^2(M; \text{ad } P)$,

$$0 = F + [\Phi, \Phi^*] = \begin{pmatrix} F_1 - \phi\phi^* & 0 \\ 0 & -F_1 + \phi\phi^* \end{pmatrix}$$

and in terms of the reducible connection on V ,

$$F(L) = F_1 + \frac{1}{2}F(\Lambda^2V).$$

However, the degree of L is given by

$$\begin{aligned}
 \deg(L) &= \frac{i}{\pi} \int_M F_1 + \frac{1}{2} F(\Lambda^2 V) \\
 &= \frac{i}{\pi} \int_M \phi \phi^* + \frac{1}{2} \deg(\Lambda^2 V) \\
 &= \frac{i}{\pi} \int \text{Tr } \Phi \Phi^* + \frac{1}{2} \deg(\Lambda^2 V) \\
 &= \frac{\mu}{\pi} + \frac{1}{2} \deg(\Lambda^2 V).
 \end{aligned} \tag{7.3}$$

Hence

$$\deg L - \frac{1}{2} \deg(\Lambda^2 V) = \mu/\pi > 0.$$

Thus μ is of the form

$$\mu = \pi(d - \frac{1}{2})$$

for an integer $d \geq 1$. By stability (see Remark (3.12)),

$$\deg(L^2 \otimes \Lambda^2 V^*) \leq 2g - 2$$

and so

$$d - \frac{1}{2} = \deg L - \frac{1}{2} \deg \Lambda^2 V \leq g - 1;$$

hence $d \leq g - 1$.

(iii) The non-degenerateness of the critical submanifolds follows from their description as the fixed point set of a circle action. Also (see [5, 21]), the *index* of the critical submanifold Y is equal to the real rank of the subbundle N^- of the normal bundle of Y on which the holomorphic circle action acts with negative weights.

In the case of $\mu^{-1}(0)$ this is necessarily zero, since 0 is an absolute minimum for μ . We have already seen that the critical submanifold is the moduli space of stable bundles.

(iv) From Theorem (4.3) and the above description of the critical set $\mu^{-1}((d - \frac{1}{2})\pi)$, this is diffeomorphic to the moduli space of stable pairs (V, Φ) where $V \cong L \oplus L^*(1)$, L being a holomorphic line bundle of degree d , and $O(1)$ a fixed line bundle of degree 1, and Φ determined by $\phi \in H^0(M; L^{-2}K(1))$.

The zero set of ϕ is thus a positive divisor of degree $2g - 2d - 1$ on M . Conversely, given a positive divisor of degree $2g - 2d - 1$ in M , we obtain a holomorphic line bundle U of degree $2d$, with a section of $U^{-1}K(1)$ vanishing on the divisor. There are 2^{2g} choices of line bundle L of degree d such that $L^2 \cong U$.

However, each such choice gives a stable pair (V, Φ) . Moreover, since $\deg L > \deg(L^* \Lambda^2 V)$, L is uniquely determined by the complex structure of V : it is the canonical subbundle of a non-semi-stable bundle. Thus this is the only choice for the complex structure of V .

The section ϕ is determined up to a non-zero constant multiple by its divisor, but the diagonal action of \mathbb{C}^* on V ,

$$\lambda(a, b) = (\lambda a, \lambda^{-1} b),$$

takes ϕ to $\lambda^2 \phi$ and so (V, Φ) , $(V, \lambda^2 \Phi)$ are in the same orbit under the group of automorphisms of V and hence the corresponding solutions of the self-duality equation are, from Theorem (4.3), gauge equivalent.

We have shown, then, that the critical set $\mu^{-1}((d - \frac{1}{2})\pi)$ is isomorphic to a covering of the space of positive divisors on M of degree $2g - 2d - 1$, or, in other words, to a covering of the $(2g - 2d - 1)$ -fold symmetric product of the surface M with itself.

To calculate the index of the critical submanifolds, we make use of the holomorphic symplectic structure on \mathcal{M} determined by the complex symplectic form ω of (6.5), or alternatively, in terms of the hyperkähler structure on \mathcal{M} , the form $\omega_2 + i\omega_3$ which is holomorphic with respect to the complex structure I .

By the definition (6.8) the submanifolds of $\mathcal{A} \times \Omega$ obtained by fixing $A \in \mathcal{A}$ and varying only $\Phi \in \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$ are isotropic with respect to ω .

From Theorem (4.3), the open subset $U \subset H^0(M; \text{End}_0 V \otimes K)$ of Higgs fields Φ which are stable with respect to the complex structure d''_A is acted upon freely by the group $\text{Aut}_0 V$ of holomorphic automorphisms of V of determinant 1, modulo ± 1 , and the quotient is a complex submanifold of \mathcal{M} . Its dimension is

$$\begin{aligned} & \dim_{\mathbb{C}} H^0(M; \text{End}_0 V \otimes K) - \dim_{\mathbb{C}} \text{Aut}_0 V \\ &= \dim_{\mathbb{C}} H^1(M; \text{End}_0 V) - \dim_{\mathbb{C}} H^0(M; \text{End}_0 V) \quad (\text{by Serre duality}) \\ &= 3g - 3 \quad (\text{by the Riemann-Roch theorem}). \end{aligned}$$

Hence, through each point of \mathcal{M} , there passes a $(3g - 3)$ -dimensional isotropic complex submanifold consisting of the equivalence classes under $\text{Aut}_0 V$ of stable Higgs fields Φ . The submanifold is clearly invariant under the circle action. Since $\dim_{\mathbb{C}} \mathcal{M} = 6g - 6$, this is the maximal dimension for an isotropic submanifold which is thus a *Lagrangian* submanifold. Consequently \mathcal{M} is foliated by Lagrangian submanifolds. We shall deal more closely with the symplectic structure of \mathcal{M} in § 8. For the moment we note that we have found in each tangent space T_m to $m \in \mathcal{M}$, a distinguished $(3g - 3)$ -dimensional subspace $W \subset T_m$, isotropic with respect to ω and such that

$$W \cong H^0(M; \text{End}_0 V \otimes K) / H^0(M; \text{End}_0 V) \cdot \Phi,$$

where $H^0(M; \text{End}_0 V) \cdot \Phi$ denotes the subspace of Higgs fields of the form $[\psi, \Phi]$, for $\psi \in H^0(M; \text{End}_0 V)$.

Now let m be a fixed point of the circle action, so that

$$\begin{aligned} V &\cong L \oplus L^*(1), \\ \Phi &= \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}, \quad \text{where } \phi \in H^0(M; L^{-2}(1)K). \end{aligned}$$

Then the circle action arises from the gauge transformation

$$g(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

Now $\text{End}_0 V \cong L^{-2}(1) \oplus L^2(-1) \oplus O$ as a holomorphic vector bundle, and $g(\theta)$ acts, relative to this decomposition, as $(e^{i\theta}, e^{-i\theta}, 1)$. Thus the subspace $H^0(M; \text{End}_0 V \otimes K)$ on which $g(\theta)$ acts with negative weight is

$$H^0(M; L^2(-1)K).$$

Since $\text{deg}(L^2(-1)) = 2d - 1 > 0$, we have

$$\dim_{\mathbb{C}} H^0(M; L^2(-1)K) = g + 2d - 2. \tag{7.4}$$

Since the circle acts with only these weights on $H^0(M; \text{End}_0 V)$ and acts on Φ as $e^{i\theta}$, there are no negative weights in $H^0(M; \text{End}_0 V) \cdot \Phi$.

Now since W is isotropic,

$$T_m/W \cong W^*. \tag{7.5}$$

The symplectic form ω is not, however, invariant under the circle action, but from its definition (6.5), it transforms as

$$\omega \rightarrow e^{i\theta}\omega.$$

Consequently, the weights $(e^{i\theta}, e^{-i\theta}, 1)$ of the action on W become

$$e^{i\theta}(e^{-i\theta}, e^{i\theta}, 1) = (1, e^{2i\theta}, e^{i\theta})$$

on W^* . In particular, there are no negative weights; hence (7.4) gives the dimension of the negative subspace of T_m . Note that from this description the subspace of T_m on which the circle acts trivially has dimension

$$\begin{aligned} \dim H^0(M; K) - \dim H^0(M; L^2(-1)) + \dim H^0(M; L^{-2}(1)K) - \dim H^0(M; O) \\ = g - (2d - g) - 1 = 2g - 2d - 1, \end{aligned}$$

by Serre duality and the Riemann–Roch theorem. This of course checks with the dimension of the critical submanifold. We have thus calculated the indices of the critical points, which are $2(g + 2d - 2)$.

Using this analysis we now prove the following theorem.

THEOREM (7.6). *Let \mathcal{M} be the moduli space of stable pairs (V, Φ) where V is a rank-2 bundle of odd degree over a Riemann surface M of genus $g > 1$, and $\Phi \in H^0(M; \text{End}_0 V \otimes K)$. Then*

- (i) \mathcal{M} is non-compact,
- (ii) \mathcal{M} is connected and simply connected,
- (iii) the Betti numbers b_i of \mathcal{M} vanish for $i > 6g - 6$,
- (iv) the Betti numbers are given by

$$\begin{aligned} \sum_{i=1}^{6g-6} b_i t^i = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}}{4(1-t^2)(1-t^4)} \\ \times \{(1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}\} - (g-1)t^{4g-3} \frac{(1+t)^{2g-2}}{(1-t)} \\ + 2^{2g-1}t^{4g-4} \{(1+t)^{2g-2} - (1-t)^{2g-2}\}. \end{aligned}$$

Proof. (i) The highest critical value of μ is, from Proposition (7.1), when $d = g - 1$. The index is then $6g - 8$ and the nullity 2 from (7.1)(iv). However, since $\dim_{\mathbb{R}} \mathcal{M} = 12g - 12$, μ cannot be a maximum. Hence μ has no maximum at all, so \mathcal{M} is non-compact.

(ii) From [5] the minimum $\mu^{-1}(0)$ is connected and simply connected. Since the indices of all critical points are even, it follows as a standard result of Morse theory that \mathcal{M} itself is connected and simply connected.

(iii) To calculate the Betti numbers of \mathcal{M} , we use the fact [5, 21] that the Morse function arising from a circle action on a Kähler manifold is perfect. This

implies that if $P_t(M) = \sum b_i t^i$ is the Poincaré polynomial of a manifold, then

$$P_t(M) = \sum_N t^{\lambda_N} P_t(N), \tag{7.7}$$

where the summation is over the critical submanifolds N , with index λ_N .

From (7.7) we see that

$$P_t(\mathcal{M}) = \sum_{d=1}^{g-1} t^{2g+4d-4} P_t(N_d) + P_t(N_0),$$

where N_d are the critical submanifolds. Since $\dim_{\mathbb{R}} N_0 = 6g - 6$ and $\dim_{\mathbb{R}} N_d = 4g - 4d - 2$, it is clear that $P_t(\mathcal{M})$ is a polynomial of degree $6g - 6$ and hence $b_i = 0$ if $i > 6g - 6$.

To compute the Betti numbers more explicitly, we use the result of Newstead [29] or Atiyah and Bott [5] for N_0 the modulo space of stable bundles of rank-2, odd degree, and fixed determinant. The Poincaré polynomial is there shown to be

$$P_t(N_0) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}. \tag{7.8}$$

The other critical submanifolds are coverings of symmetric products of the Riemann surface M . The Poincaré polynomial of a symmetric product was computed by Macdonald [23]:

$$P_t(S^n M) \text{ is equal to the coefficient of } x^n \text{ in } \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}. \tag{7.9}$$

Consider now the 2^{2g} -fold covering $\tilde{S}^n M$ of $S^n M$, induced by the map

$$j: S^n M \rightarrow \text{Jac}(M).$$

This covering is a principal \mathbb{Z}_2^{2g} bundle over $S^n M$, and so by the Leray sequence,

$$H^p(\tilde{S}^n M, \mathbb{R}) \cong \bigoplus_{i=1}^{2^{2g}} H^p(S^n M; \mathcal{L}_i), \tag{7.10}$$

where \mathcal{L}_i is the sheaf of locally constant sections of the flat line bundle of order 2 over $S^n M$ corresponding to the \mathbb{Z}_2 -homomorphism $\alpha_i: \mathbb{Z}_2^{2g} \rightarrow \mathbb{Z}_2$, for $1 \leq i \leq 2^{2g}$.

Now let M^n denote the n -fold Cartesian product of M and $f: M^n \rightarrow S^n M$ the projection map. If $\xi \in H^1(\text{Jac}(M), \mathbb{Z}) \cong H^1(M, \mathbb{Z})$ then (see [23]),

$$f^* j^* \xi = \xi \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \xi \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes \xi$$

as an element of $H^1(M^n, \mathbb{Z}) \cong \bigoplus_{i=1}^n H^1(M, \mathbb{Z})$ in the Künneth decomposition.

Thus given a flat line bundle of order 2 over $\text{Jac}(M)$, this corresponds to an element in $H^1(\text{Jac}(M), \mathbb{Z}_2) \cong H^1(M, \mathbb{Z}_2)$ and hence a line bundle L_i of order 2 over M which pulls back to M^n to give the bundle

$$L_i \otimes L_i \otimes L_i \otimes \dots \otimes L_i \text{ over } M^n = M \times M \times \dots \times M. \tag{7.11}$$

From (7.10) we need to compute $H^p(S^n M; \mathcal{L}_i)$ which is the S_n -invariant part of $H^p(M^n; \mathcal{L}_i)$ where \mathcal{L}_i is the sheaf of locally constant sections of the flat line bundle in (7.11). If L_i is trivial, this is just the ordinary cohomology of $S^n M$ which is given by Macdonald's formula, so suppose L_i is non-trivial. Then

$$H^0(M; \mathcal{L}_i) = 0$$

and hence by duality

$$H^2(M ; \mathcal{L}_i) = 0.$$

By the index theorem, it follows that

$$\dim H^1(M ; \mathcal{L}_i) = 2g - 2.$$

Now consider the fibration

$$M \rightarrow M^n \rightarrow M^{n-1}$$

and apply the Leray spectral sequence to the sheaf \mathcal{L}_i of constant sections of the line bundle in (7.11). Since only $H^1(M, \mathcal{L}_i)$ is non-zero, the spectral sequence degenerates and

$$H^p(M^n ; \mathcal{L}_i) \cong H^1(M ; \mathcal{L}_i) \otimes H^{p-1}(M^{n-1} ; \mathcal{L}_i),$$

and hence by induction

$$H^p(M^n ; \mathcal{L}_i) = \begin{cases} 0 & \text{if } p \neq n, \\ \otimes^n H^1(M ; \mathcal{L}_i) & \text{if } p = n. \end{cases} \tag{7.12}$$

Since elements of $H^1(M^n ; \mathcal{L}_i)$ anti-commute, the symmetric part of $H^n(M^n ; \mathcal{L}_i)$, which is isomorphic to $H^n(S^n M ; \mathcal{L}_i)$, is the alternating part of $\otimes^n H^1(M ; \mathcal{L}_i)$, that is,

$$H^n(S^n M ; \mathcal{L}_i) \cong \Lambda^n H^1(M ; \mathcal{L}_i).$$

Consequently from (7.1) we obtain the following Betti numbers of $\bar{S}^n M$:

$$\begin{aligned} b_k(\bar{S}^n M) &= b_k(S^n M) \quad \text{if } k \neq n, \\ b_n(\bar{S}^n M) &= b_n(S^n M) + (2^{2g} - 1) \binom{2g - 1}{n}. \end{aligned} \tag{7.13}$$

Thus from (7.1) and (7.7) the contribution to the Poincaré polynomial of \mathcal{M} from the non-minimal critical submanifolds is

$$\sum_{d=1}^{g-1} t^{2(g+2d-2)} P_t(S^{2g-2d-1} M) + (2^{2g} - 1) \sum_{d=1}^{g-1} \binom{2g - 2}{2g - 2d - 1} t^{4g+2d-5}.$$

The second term is easily evaluated to be

$$\frac{1}{2}(2^{2g} - 1)t^{4g-4}[(1+t)^{2g-2} - (1-t)^{2g-2}]. \tag{7.14}$$

The first term, using the Poincaré polynomial for a symmetric product (7.9), gives the coefficient of x^{2g} in

$$\sum_{d=1}^{g-1} t^{2(g+2d-2)} x^{2d+1} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}.$$

This is clearly the coefficient of x^{2g} in the infinite sum

$$\sum_{d=1}^{\infty} t^{2(g+2d-2)} x^{2d+1} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}$$

which equals

$$\frac{(1+xt)^{2g} t^{2g} x^3}{(1-x)(1-xt^2)(1-x^2 t^4)}.$$

We therefore require the residue at $x = 0$ of

$$f(x) = \frac{(1+xt)^{2g}t^{2g}}{(1-x)(1-xt^2)^2(1+xt^2)x^{2g-2}}. \quad (7.15)$$

The rational function $f(x)$ has, apart from $x = 0$, simple poles at $x = 1$ and $x = -t^{-2}$ and a double pole at $x = t^{-2}$. Moreover, as $x \rightarrow \infty$,

$$f(x) \sim t^{4g-6}/x^2.$$

Hence, by Cauchy's residue theorem, the required residue is also

$$\sum_{x=1, \pm t^{-2}} \text{Res} \frac{t^{2g}(1+xt)^{2g}}{(x-1)(xt^2-1)^2(xt^2+1)x^{2g-2}}.$$

At the simple pole $x = 1$, we obtain

$$\text{Res}_{x=1} = \frac{t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}. \quad (7.16)$$

At the simple pole $x = -t^{-2}$, we obtain

$$\text{Res}_{x=-t^{-2}} = -\frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)}. \quad (7.17)$$

At the double pole $x = t^{-2}$, we obtain

$$\begin{aligned} \text{Res}_{x=t^{-2}} &= \frac{t^{4g-4}(1+t)^{2g}}{2(1-t^2)} \left\{ \frac{2g}{(1+t)} - \frac{1}{2} - \frac{1}{(1-t^2)} - (2g-2) \right\} \\ &= \frac{t^{4g-4}(1+t)^{2g}}{2(1-t^2)} \left\{ \frac{(2-2g)t}{(1+t)} + \frac{t^2-4t+1}{2(1-t^2)} \right\}. \end{aligned} \quad (7.18)$$

Hence, from (7.8), (7.14), (7.16), (7.17), and (7.18),

$$\begin{aligned} P_t(\mathcal{M}) &= \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)} \\ &\quad - \frac{t^{4g-3}(1+t)^{2g-2}(g-1)}{(1-t)} + \frac{t^{4g-4}(1+t)^{2g-2}(t^2-4t+1)}{4(1-t)^2} \\ &\quad + \frac{1}{2}(2^{2g}-1)t^{4g-4}\{(1+t)^{2g-2} - (1-t)^{2g-2}\}, \end{aligned}$$

which reduces to

$$\begin{aligned} P_t(\mathcal{M}) &= \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}}{4(1-t^2)(1-t^4)} \{(1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}\} \\ &\quad - (g-1)t^{4g-3} \frac{(1+t)^{2g-1}}{(1-t)} + 2^{2g-2}t^{4g-4}\{(1+t)^{2g-2} - (1-t)^{2g-2}\}. \end{aligned}$$

EXAMPLE. In the case of a surface of genus 2 we simply have

$$\begin{aligned} P_t(\mathcal{M}) &= P_t(N_0) + t^4 P_t(N_1) \\ &= 1 + t^2 + 4t^3 + t^4 + t^6 + t^4(1 + 34t + t^2) \\ &= 1 + t^2 + 4t^3 + 2t^4 + 34t^5 + 2t^6. \end{aligned}$$

8. The symplectic structure of the moduli space

In the course of the proof of Proposition (7.1) we made use of the holomorphic symplectic structure on \mathcal{M} , the moduli space of stable pairs (V, Φ) . We saw there that \mathcal{M} has a natural polarization, i.e. a foliation by Lagrangian submanifolds each of which is obtained by fixing the equivalence class of the complex structure and allowing Φ to vary.

In the case where V is a stable vector bundle, there are no non-trivial automorphisms, and the leaf of the foliation is just the vector space $H^0(M; \text{End}_0 V \otimes K)$. This is canonically dual by Serre duality to the tangent space $H^1(M; \text{End}_0 V)$ of the moduli space \mathcal{N} of stable vector bundles of fixed determinant and so we find the cotangent bundle $T^*\mathcal{N}$ embedded as an open set in \mathcal{M} . It is not difficult to see that the complex symplectic form ω is the canonical form on the cotangent bundle. It is however hard to visualize the symplectic manifold obtained by adjoining further leaves of the foliation and extending the symplectic form.

We prefer here to consider the symplectic structure from another point of view, producing a polarization on an open set which is transverse to the one described above. For this we consider the gauge-invariant map

$$(A, \Phi) \rightarrow \det \Phi \in H^0(M; K^2),$$

which clearly defines a holomorphic map

$$\det: \mathcal{M} \rightarrow H^0(M; K^2).$$

THEOREM (8.1). *Let \mathcal{M} be the moduli space of stable pairs (V, Φ) on a Riemann surface M of genus $g > 1$, where V is a vector bundle of rank 2 and odd degree, with fixed determinant and $\Phi \in H^0(M; \text{End}_0 V \otimes K)$. Then the map*

$$\det: \mathcal{M} \rightarrow H^0(M; K^2)$$

satisfies the following properties:

- (i) *det is proper;*
- (ii) *det is surjective;*
- (iii) *if $\alpha_1, \dots, \alpha_{3g-3}$ is a basis for $H^0(M; K^2)^* \cong H^1(M; K^{-1})$ then the functions $f_i = \alpha_i(\det)$ commute with respect to the Poisson bracket determined by the holomorphic symplectic structure of \mathcal{M} ;*
- (iv) *if $q \in H^0(M; K^2)$ is a quadratic differential with simple zeros, then $\det^{-1}(q)$ is biholomorphically equivalent to the Prym variety of the double covering of M determined by the 2-valued differential \sqrt{q} .*

REMARKS. Since $\dim_{\mathbb{C}} \mathcal{M} = 6g - 6 = 2 \dim_{\mathbb{C}} H^0(M; K^2)$, the third part of the theorem says that \mathcal{M} is a completely integrable Hamiltonian system. The Hamiltonian vector fields corresponding to the functions f_i then define linearly independent commuting vector fields on each regular fibre. Since \det is proper, each component of a regular fibre is thus automatically a complex torus. Part (iv) identifies this torus as the Prym variety. Recall that if \tilde{M} is a double covering of M with involution $\sigma: \tilde{M} \rightarrow \tilde{M}$ interchanging the branches, then the Prym variety is the subvariety of the Jacobian of \tilde{M} on which σ acts as -1 .

Proof. (i) From (4.18) we have the inequality

$$\|\Phi\|_{L^2}^4 \leq c_1 + c_2 \int_M |\det \Phi|^2$$

and from (4.17),

$$\|F\|_{L^2}^2 = \|[\Phi, \Phi^*]\|_{L^2}^2 \leq c \|\Phi\|_{L^2}^2.$$

Hence if $\|\det \Phi\| \leq M$ in the finite-dimensional vector space $H^0(M; K^2)$, then we have an L^2 bound on $F(A)$ for all corresponding solutions of the self-duality equations. By Uhlenbeck's theorem, as used in Theorem (4.3), every sequence in $\det^{-1}(B_M(0))$ has a convergent subsequence and hence \det is proper.

(ii) The Poisson bracket of two functions on a symplectic manifold is defined by

$$\{f_1, f_2\} = -X_1 \cdot f_2 = -df_2(X_1),$$

where X_1 is the Hamiltonian vector field corresponding to f_1 , that is,

$$i(X_1)\omega = df_1.$$

Now let $\alpha_i \in H^1(M; K^{-1})$ be represented by forms $\beta_i \in \Omega^{0,1}(M; K^{-1})$. Then $2\beta_i \det \Phi = -\beta_i \operatorname{Tr} \Phi^2$ is an element of $\Omega^{0,1}(M; K)$ and the functions $f_i = 2\alpha_i(\det)$ may be expressed as

$$f_i = - \int_M \beta_i \operatorname{Tr} \Phi^2. \tag{8.2}$$

The complex symplectic form on $\mathcal{A} \times \Omega$ is defined from (6.5) as

$$\omega((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = \int_M \operatorname{Tr}(\Phi_2 \Psi_1 - \Phi_1 \Psi_2). \tag{8.3}$$

As noted in § 6, the moment map for the symplectic action of the group \mathcal{G}^c of complex automorphisms of $\operatorname{ad} P \otimes \mathbb{C}$ with respect to this complex symplectic structure is

$$\mu(A, \Phi) = d_A'' \Phi \in \Omega^{1,1}(M; \operatorname{ad} P \otimes \mathbb{C}).$$

The form ω is degenerate along the orbits under \mathcal{G}^c , and for this reason descends (the Marsden-Weinstein quotient) to a form on the quotient, which is the symplectic form under consideration here.

We may take then a tangent vector $(\dot{A}, \dot{\Phi})$ in $\mathcal{A} \times \Omega$, tangent to $d_A'' \Phi = 0$ to represent a tangent vector X_i on \mathcal{M} , and use the symplectic form (8.3) for computations. Thus,

$$\begin{aligned} df_i(\dot{B}, \dot{\Psi}) &= i(X_i)\omega(\dot{B}, \dot{\Psi}) \\ &= \int_M \operatorname{Tr}(\dot{\Psi}\dot{A} - \dot{\Phi}\dot{B}). \end{aligned} \tag{8.4}$$

But from (8.2),

$$df_i(\dot{B}, \dot{\Psi}) = -2 \int_M \beta_i \operatorname{Tr}(\Phi\dot{\Psi}).$$

Hence from (8.4) we may represent the vector field X_i on \mathcal{M} by the field in

$\mathcal{A} \times \Omega$,

$$(\dot{A}, \dot{\Phi}) = (-2\beta_i \Phi, 0). \tag{8.5}$$

Since $\dot{\Phi} = 0$,

$$\{f_1, f_2\} = -df_2(X_1) = 2 \int_M \beta_2 \operatorname{Tr}(\Phi \dot{\Phi}) = 0,$$

and so the functions Poisson-commute.

(iv) Let (V, Φ) be a stable pair with

$$\det \Phi = -q \in H^0(M; K^2)$$

and suppose q has simple zeros. In the total space of the canonical bundle K_M of M we consider the subvariety

$$\tilde{M} = \{a_x \in K_x \mid a_x^2 = q(x) \in K_x^2\}.$$

If q has simple zeros, \tilde{M} is a non-singular Riemann surface with an involution:

$$\sigma: \tilde{M} \rightarrow \tilde{M}, \quad \sigma(a_x) = -a_x.$$

By the Riemann–Hurwitz formula, the genus of \tilde{M} is

$$\tilde{g} = 4g - 3. \tag{8.6}$$

Let $p: \tilde{M} \rightarrow M$ denote the projection, a double covering branched over the zeros of q . Then by the definition of \tilde{M} , \sqrt{q} defines a holomorphic section of p^*K_M on \tilde{M} . Pulling back the vector bundle V to \tilde{M} , we have Φ defining

$$\tilde{\Phi} \in H^0(\tilde{M}; p^*(\operatorname{End}_0 V \otimes K_M)).$$

However, since

$$\det \Phi = -q = -(\sqrt{q})^2,$$

we have two rank-1 subbundles of p^*V defined by

$$L_1 \subseteq \ker(\tilde{\Phi} + \sqrt{q}), \quad L_2 \subseteq \ker(\tilde{\Phi} - \sqrt{q}), \tag{8.7}$$

which are clearly interchanged by the involution σ . The two subbundles coincide where $q = 0$; hence the holomorphic homomorphism

$$\Lambda: L_1 \otimes L_2 \rightarrow \Lambda^2 p^*V$$

vanishes at the zeros of \sqrt{q} with multiplicity 1. Thus

$$L_1 \otimes L_2 \cong p^*(\Lambda^2 V) \otimes p^*K_M^{-1}. \tag{8.8}$$

Since L_1 and L_2 are interchanged by σ , we have from (8.8),

$$\sigma(L_1) \cong L_1^* \otimes p^*(\Lambda^2 V \otimes K_M^{-1}). \tag{8.9}$$

As $p^*(\Lambda^2 V \otimes K_M^{-1})$ is of even degree, p being a double covering, and invariant by σ , we may choose a fixed line bundle L_0 such that

$$p^*(\Lambda^2 V \otimes K_M^{-1}) \cong L_0 \otimes \sigma(L_0)$$

and then (8.9) becomes

$$\sigma(L_1 \otimes L_0^{-1}) \cong L_1^{-1} \otimes L_0$$

that is, $L_1 \otimes L_0^{-1}$ is a line bundle L of degree zero such that $\sigma(L) \cong L^*$, and this gives a point in the Prym variety of \tilde{M} . Note that

$$\dim(\text{Prym } \tilde{M}) = \tilde{g} - g = 3g - 3 \quad \text{from (8.6).}$$

Conversely, suppose L_1 is a line bundle on \tilde{M} with $\deg L_1 = \deg(\Lambda^2 V \otimes K_M^{-1})$ and such that $L_1 \otimes L_0^{-1} \in \text{Prym}(\tilde{M})$. The direct image $p_* L_1$ defines a rank-2 vector bundle on M . The fibre of this bundle at $x \in M$ is by definition

$$(p_* L_1)_x \cong O_{\tilde{M}}(L_1) / \mathcal{I}_{p^{-1}(x)}$$

where $\mathcal{I}_{p^{-1}(x)}$ is the ideal sheaf of $p^{-1}(x)$. Considering the reduction of the divisor $p^{-1}(x)$ at the branch points x , we have a natural sheaf map

$$s: (p_* L_1)^* \rightarrow \mathcal{S}$$

to a sheaf supported on the branch points. It is the direct image of the sheaf $O_{\tilde{M}}(K_M^* L_1^*) / \mathcal{I}_R$ where $R \subset \tilde{M}$ is the ramification divisor.

The kernel of this map is a locally free sheaf of rank 2, the sheaf $O_M(V^*)$ for a vector bundle V . By construction, there is a canonical inclusion of $L_1 \subset p^* V$, and also since $\mathcal{I}_{p^{-1}(x)} = \mathcal{I}_y \cdot \mathcal{I}_{\sigma y}$, an inclusion of $\sigma(L_1)$. Now $\sigma(L_1) \subset p^* V$ coincides with L_1 at the fixed points of σ : the branch points of the covering. It follows directly that since $L_1 L_0^{-1}$ is in the Prym variety,

$$p^*(\Lambda^2 V) \cong L_0 \sigma(L_0) \otimes p^* K_M$$

and so all the vector bundles V constructed this way have the same determinant bundle.

We may define

$$\tilde{\Phi}: p^* V \rightarrow p^* V \otimes p^* K_M$$

by $\tilde{\Phi}(v) = \sqrt{q}v$ if $v \in L_1$ and $\tilde{\Phi}(v) = -\sqrt{q}$ for $v \in \sigma(L_1)$, and since this is invariant by σ , it descends to $\Phi \in H^0(M; \text{End}_0 V \otimes K)$.

Now if Φ left a subbundle $L \subset V$ invariant, we would have $\Phi(v) = \theta v$ for $v \in L$ and some $\theta \in H^0(M; K)$. Hence (since $\text{Tr } \Phi = 0$) we would have $\det \Phi = -q = -\theta^2$ which is impossible if q has simple roots. Thus the Φ constructed this way gives a stable pair (V, Φ) .

The variety $\{L \in H^1(M; O^*): \sigma(L) = L^{-1}\}$ is therefore mapped biholomorphically onto the fibre $\det^{-1}(-q)$.

(ii) As a consequence of the above, we have seen that any $q \in H^0(M; K^2)$ which has simple zeros is in the image of \det . Since this is a Zariski open set, $\det(\hat{\mathcal{M}})$ is dense in $H^0(M; K^2)$. However, since \det is proper, it must be surjective.

It is instructive to consider the map \det restricted to the open set $T^* \mathcal{N}$ which corresponds to stable pairs (V, Φ) where V itself is a stable bundle since in principle we have more explicit knowledge about \mathcal{N} . We thus consider the bundle V on M defined by the direct image of a line bundle L_1 on \tilde{M} as in the theorem and ask whether V is stable.

The bundle V will be unstable if and only if there exists a line bundle $L \subset V$ such that $\deg(L) > \frac{1}{2} \deg \Lambda^2 V$. Let \tilde{L} denote the pulled back line bundle on \tilde{M} . Then

$$\sigma^* \tilde{L} = \tilde{L}. \tag{8.10}$$

Now if V is pulled back, it is described as an extension

$$0 \rightarrow L_1 \rightarrow p^*V \rightarrow L_1^* \otimes p^*(\Lambda^2 V) \rightarrow 0. \quad (8.11)$$

Consider the exact sequence of sheaves

$$0 \rightarrow \tilde{L}^* L_1 \rightarrow \tilde{L}^* p^*V \rightarrow \tilde{L}^* L_1^* \otimes p^*(\Lambda^2 V) \rightarrow 0 \quad (8.12)$$

and its corresponding long exact cohomology sequence. Since, as in (8.9),

$$\deg L_1 = \deg(\Lambda^2 V) - (2g - 2)$$

(remarking that $\tilde{M} \rightarrow M$ is a double covering) and

$$\deg \tilde{L} > \deg(\Lambda^2 V)$$

we have

$$\deg(\tilde{L}^* L_1) < -(2g - 2)$$

and so $H^0(\tilde{M}; \tilde{L}^* L_1) = 0$. Thus the inclusion $\tilde{L} \subset p^*V$ maps to a non-zero element of $H^0(\tilde{M}; \tilde{L}^* L_1^* \otimes p^*(\Lambda^2 V))$. This is a section of a line bundle on \tilde{M} of degree

$$\deg(\tilde{L}^* L_1^* \otimes p^*(\Lambda^2 V)) < (2g - 2) = \frac{1}{2}(\tilde{g} - 1)$$

and thus defines a point in the space of special divisors W_{2g-3} .

Suppose conversely that \tilde{L} is a line bundle on \tilde{M} , invariant under σ as in (8.10) and such that $\tilde{L}^* L_1^* \otimes p^*(\Lambda^2 V)$ has a non-trivial section s . From (8.12) this comes from a homomorphism from \tilde{L} to p^*V if and only if s goes to zero under the coboundary map

$$\delta: H^0(\tilde{M}; \tilde{L}^* L_1^* \otimes p^*(\Lambda^2 V)) \rightarrow H^1(\tilde{M}; \tilde{L}^* L_1)$$

which is the cup product with the extension class e in $H^1(\tilde{M}, L_1^2 \otimes p^*(\Lambda^2 V^*))$ which defines p^*V in (8.11). However, p^*V is an extension also by $L_2 = \sigma^*(L_1)$ which means that the extension class e arises from the exact sequences of sheaves:

$$0 \rightarrow L_1^2 \otimes p^*(\Lambda^2 V^*) \rightarrow L_2^* L_1 \rightarrow L_2^* L_1 \big|_D \rightarrow 0$$

where D is the fixed point set of σ . Thus

$$e = \delta_D(\sigma^*)$$

where $\sigma^* \in H^0(D; L_2^* L_1)$ is the isomorphism $L_2 = \sigma^* L_1$ restricted to the fixed points of σ , and

$$\delta_D: H^0(D; L_2^* L_1) \rightarrow H^1(\tilde{M}; L_1^2 \otimes p^*(\Lambda^2 V^*))$$

is the coboundary map for this sequence. (An analogous situation occurs in the twistor description of monopoles [16].)

It follows that $\delta(s) = 0$ if and only if $\delta_D(\sigma^* s) = 0$. Now $\sigma^* s$ is a section of $\tilde{L}^* L_2^* \otimes p^*(\Lambda^2 V)$ on D and this maps to zero if and only if it restricts from a section of the same bundle on \tilde{M} . However, since $\sigma^* \tilde{L} = \tilde{L}$ and $\sigma^* L_1 = L_2$, the action of the involution σ on s gives just such a section.

Hence if $\tilde{L}^* L_1^* \otimes p^* \Lambda^2 V$ has a non-vanishing section, there is a homomorphism from \tilde{L} to p^*V which is in fact by construction σ -invariant and hence defines a homomorphism on M from L to V .

The points in the Prym variety which define unstable bundles on M are thus of

the form

$$m^{-1}(W_{2g-3}) \cap \text{Prym } \tilde{M} \times \{1\}$$

where

$$m: \text{Prym}(\tilde{M}) \times \text{Jac}(M) \rightarrow \text{Jac}(\tilde{M})$$

is the natural multiplication map and $W_{2g-3} \subset \text{Jac}(\tilde{M})$ is the $(2g - 3)$ -dimensional space of special divisors corresponding to line bundles of degree $2g - 3$ with a non-trivial section. These unstable points are generically of codimension g in $\text{Prym}(\tilde{M})$.

Reinterpreting this information we see that the generic fibre of

$$\det: T^*\mathcal{N} \rightarrow H^0(M; K^2)$$

is non-compact and consists of removing a codimension- g subvariety from a $(3g - 3)$ -dimensional torus. We may thus regard \mathcal{M} as a fibre-wise compactification of the symplectic manifold $T^*\mathcal{N}$ with respect to the function \det .

This perhaps lends the space \mathcal{M} a more natural algebro-geometric interpretation in terms of the more well-understood moduli space \mathcal{N} of stable bundles.

The symplectic point of view may be taken further in different directions. We shall consider one more aspect here. This is the formation of ‘caustics’ by the Lagrangian submanifold $\text{Prym}(\tilde{M}) \subset \mathcal{M}$. We consider the points of $\text{Prym}(\tilde{M})$ at which the polarization of \mathcal{M} obtained by fixing the equivalence class of complex structure on V is tangential. Restricting to $T^*\mathcal{N}$ we see that these are the points at which the projection onto \mathcal{N} is not a local diffeomorphism. The projection onto \mathcal{N} gives a subspace which is called a caustic. The singularities of projections of Lagrangian submanifolds have been investigated in depth by Arnol’d [2].

For our purposes we merely wish to determine the locus of points in the Prym variety at which the polarization has a tangential part. This is equivalent to seeking a tangent vector X to \mathcal{M} , tangent to the polarization by complex structures, and such that

$$df_i(X) = 0 \quad \text{for } 1 \leq i \leq 3g - 3,$$

where the f_i are the functions in (8.1) which define the integrable system. Since $\det \Phi = -\frac{1}{2} \text{Tr } \Phi^2$, this is more invariantly where $\text{Tr}(\Phi\psi) = 0$ for some $\psi \in H^0(M; \text{End}_0 V \otimes K)$.

Pulling back to \tilde{M} , we have $p^*V \cong L_1 \oplus L_2$ outside the fixed points of σ and with respect to this decomposition,

$$\Phi = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}.$$

Hence, if

$$\psi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

then $\text{Tr}(\Phi\psi) = 0$ if and only if $a = 0$. This implies that

$$\psi(L_1) \subseteq L_2 K_M, \quad \psi(L_2) \subseteq L_1 K_M. \tag{8.13}$$

If this is true on $\tilde{M} \setminus D$, it will of course be true elsewhere. Thus, there must exist a non-zero section of $L_2^* L_1 K_M$ or of $L_1^* L_2 K$. In fact, applying σ , we see that such sections occur at the same time.

Now since $L_2 \cong \sigma^*(L_1)$, we have $\deg(L_2^*L_1) = 0$ and by the construction of \tilde{M} , $K_{\tilde{M}} \cong p^*(K_M^2)$. Thus $L_2^*L_1K_M$ is a line bundle of degree $\bar{g} - 1$ on \tilde{M} with a holomorphic section. Thus $U = L_1L_0^* \in \text{Prym}(\tilde{M})$ lies in $f^{-1}(W_{\bar{g}-1})$ under

$$f: \text{Prym}(\tilde{M}) \rightarrow \text{Jac}^{\bar{g}-1}(\tilde{M})$$

defined by $f(U) = U^2K_M$. Thus in the moduli space \mathcal{M} of $\text{SO}(3)$ solutions, this subvariety of the torus is the intersection of *the theta divisor of \tilde{M}* with the Prym variety.

Conversely, suppose there is a non-vanishing section s of $L_2^*L_1K_M$ and hence a corresponding section σ^*s of $L_1^*L_2K_M$. Consider the homomorphism

$$\alpha: L_1 \oplus L_2 \rightarrow p^*V, \quad \alpha(l_1, l_2) = l_1 + l_2,$$

induced by the inclusions of L_1, L_2 in p^*V . There is an induced homomorphism

$$S_\alpha^2: S^2(L_1 \oplus L_2) \rightarrow p^*S^2V$$

and consequently a homomorphism

$$\beta: \text{End}_0(L_1 \oplus L_2) \cong S^2(L_1 \oplus L_2)L_1^*L_2^* \rightarrow p^*\text{End}_0 V \otimes p^*\Lambda^2V \otimes L_1^*L_2^*.$$

The homomorphism β vanishes at the fixed point set of σ which is a divisor D of K_M . If therefore ϕ is a section of $\text{End}_0(L_1 \oplus L_2)K_M$, then $\beta(\phi)$ is a section of

$$p^*\text{End}_0 V \otimes p^*\Lambda^2V \otimes L_1^*L_2^* \otimes K_M \cong p^*\text{End}_0 V \otimes K_M^2 \quad \text{by (8.8).}$$

This vanishes on D and hence defines a section of $p^*(\text{End}_0 V \otimes K_M)$. Putting

$$\phi = s + \sigma^*s \in H^0(\tilde{M}; \text{End}_0(L_1 \oplus L_2)K_M)$$

we obtain a σ -invariant section of $p^*(\text{End}_0 V \otimes K_M)$ and hence a section ψ of $\text{End } V \otimes K$ on M which by construction satisfies $\text{Tr}(\Phi\psi) = 0$.

Thus *the theta divisor is the locus of the singular points with respect to the projection onto \mathcal{N}* , or more generally the foliation by complex structures of V .

9. The other complex structures

In § 6 we showed that the moduli space \mathcal{M} possesses a natural hyperkähler metric and hence complex structures I, J, K with respect to all of which the metric is Kählerian. In fact if \mathbf{x} is a unit vector in \mathbb{R}^3 , then

$$(x_1I + x_2J + x_3K)^2 = -1$$

and we have a 2-sphere of complex structures.

All our investigations so far have, however, used just one of these structures, I , which endowed \mathcal{M} with the natural complex structure of the moduli space of stable pairs (V, Φ) . We shall now consider the other complex structures on \mathcal{M} .

PROPOSITION (9.1). *Let \mathcal{M} be the moduli space of solutions to the self-duality equations on a rank-2 vector bundle of odd degree and fixed determinant over a compact Riemann surface of genus at least 2. Then*

- (i) *all the complex structures of the hyperkählerian family other than $\pm I$ are equivalent,*

- (ii) with respect to each such structure, \mathcal{M} is a Stein manifold,
- (iii) \mathcal{M} has no non-constant bounded holomorphic functions.

Proof. (i) First recall that the $U(1)$ action on \mathcal{M} defined by $(A, \Phi) \rightarrow (A, e^{i\theta}\Phi)$ preserves the symplectic form ω_1 but acts on the form $\omega_2 + i\omega_3$ as $e^{i\theta}(\omega_2 + i\omega_3)$. If X is the vector field generated by the action, then

$$\mathcal{L}_X\omega_2 = \omega_3, \quad \mathcal{L}_X\omega_3 = -\omega_2. \tag{9.2}$$

Now define a vector field \bar{X} on $\mathcal{M} \times S^2$ by taking the product action of the circle on \mathcal{M} and S^2 , regarding S^2 as the space of covariant constant 2-forms $x_1\omega_1 + x_2\omega_2 + x_3\omega_3$ of unit length acted on according to (9.2). Alternatively it is the space of complex structures $\{I_{\mathbf{x}} = x_1I + x_2J + x_3K : \mathbf{x} \in S^2\}$ of the hyperkähler metric.

The product manifold $\mathcal{M} \times S^2$ has a natural complex structure $\bar{J} = (I_{\mathbf{x}}, I_{S^2})$ which is integrable (this is the *twistor space* of the hyperkählerian structure [17, 18, 31]) and the circle action preserves the complex structure.

The action projects under the holomorphic projection

$$p_2: \mathcal{M} \times S^2 \rightarrow S^2 \cong \mathbb{C}P^1$$

to the standard rotation leaving $\pm I$ fixed. If we can show that the holomorphic vector field on $\mathcal{M} \times S^2$ generated by \bar{X} extends to a \mathbb{C}^* action, then this must cover the \mathbb{C}^* action on $\mathbb{C}P^1$ which acts transitively on the complement of $\pm I$ and hence will carry \mathcal{M} with any complex structure apart from $\pm I$ to any other, which is what is required for (i). We must therefore show that the vector field $\bar{J}\bar{X}$ is complete.

Without loss of generality, consider an integral curve $\gamma(t)$ of the vector field which projects as t increases to an integral curve of the vector field on S^2 pointing to $+I$. Now

$$\frac{d\gamma(t)}{dt} = \bar{J}\bar{X}$$

and

$$\frac{d\alpha(t)}{dt} \stackrel{\text{def}}{=} \frac{d}{dt}(p_1\gamma(t)) = I_{\mathbf{x}(t)}X$$

where $p_1: \mathcal{M} \times S^2 \rightarrow \mathcal{M}$ is the projection. The curve $\alpha(t)$ in \mathcal{M} satisfies

$$g\left(\frac{d}{dt}, IX\right) = g(I_{\mathbf{x}(t)}X, IX) = x_1 \|X\|^2, \tag{9.3}$$

and since $x_1(t)$ is increasing by assumption, we may assume for $t > t_0$,

$$\|X\|^2 \geq g\left(\frac{d}{dt}, IX\right) \geq c \|X\|^2. \tag{9.4}$$

But from §§ 6 and 7, if $\mu = \|\Phi\|_{L^2}^2$, then

$$IX = -\frac{1}{2} \text{grad } \mu.$$

Hence from (9.4) we obtain, along the curve,

$$c_1 \|X\|^2 \leq \frac{d\mu}{dt} \leq c_2 \|X\|^2. \tag{9.5}$$

Recall now from the proof of (6.3) that if $l(t_0, t_1)$ is the length of the curve from t_0 to t_1 , then

$$\|\Phi(t_0) - \Phi(t_1)\|_{L^2} \leq l(t_0, t_1)$$

and thus

$$|\mu^{\frac{1}{2}}(t_0) - \mu^{\frac{1}{2}}(t_1)| \leq l(t_0, t_1),$$

and hence

$$\left| \frac{d}{dt} (\mu^{\frac{1}{2}}) \right| \leq \frac{ds}{dt}. \quad (9.6)$$

Now

$$\left(\frac{ds}{dt} \right)^2 = g \left(\frac{d}{dt}, \frac{d}{dt} \right) = g(I_x X, I_x X) = \|X\|^2. \quad (9.7)$$

Thus from (9.5) and (9.6) we have

$$\left| \frac{d}{dt} (\mu^{\frac{1}{2}}) \right| \leq \|X\| \leq K \left| \frac{d\mu}{dt} \right|^{\frac{1}{2}}. \quad (9.8)$$

Therefore

$$\mu^{-1} \left(\frac{d\mu}{dt} \right)^2 \leq 4K \left| \frac{d\mu}{dt} \right| \quad \text{and} \quad \left| \frac{d\mu}{dt} \right| \leq 4K\mu.$$

Thus $\mu \leq ce^{4Kt}$. Consequently if $\alpha(t)$, with $t \in [0, t_1)$, is a maximal integral curve, then μ is bounded on $\alpha(t)$, but since μ is proper from (7.1), $\alpha(t)$ has a convergent subsequence as $t \rightarrow t_1$. From (9.7) and (9.8), $d\alpha/dt$ is bounded as $t \rightarrow t_1$ and hence $\alpha(t)$ converges as $t \rightarrow t_1$. Moreover from the differential equation $d\alpha/dt$ converges too. Hence, integrating the vector field in a neighbourhood of $\lim_{t \rightarrow t_1} \alpha(t)$, we obtain a contradiction to maximality. Thus every integral curve may be extended to all values of t and the vector field is complete.

(ii) From (i) it is enough to consider the complex structure J .

Recall that $f = -\frac{1}{2}\mu$ is the moment map with respect to the symplectic structure ω_1 of the $U(1)$ action. Hence

$$i(X)\omega_1 = df.$$

Now for a vector field Y ,

$$df(JY) = -i(d'_j - d''_j)f(Y),$$

whence

$$\begin{aligned} (d'_j - d''_j)f(Y) &= idf(JY) \\ &= i(i(X)\omega_1)(JY) \\ &= ig(IX, JY) \\ &= ig(KX, Y) \\ &= i(i(X)\omega_3)(Y). \end{aligned}$$

In other words,

$$(d'_j - d''_j)f = i(i(X)\omega_3). \quad (9.9)$$

Thus

$$-2d'_j d''_j f = id(i(X)\omega_3) = i\mathcal{L}_X \omega_3.$$

But from (9.2), $\mathcal{L}_X \omega_3 = -\omega_2$, and hence

$$\omega_2 = -2id'_j d''_j f. \tag{9.10}$$

Thus what was a moment map for the Kähler form ω_1 is now a *Kähler potential*.

Since f is proper and ω_2 is a Kähler form, f provides a strictly plurisubharmonic exhaustion function for \mathcal{M} with respect to the complex structure J . Thus \mathcal{M} is a Stein manifold.

(iii) Since a hyperkähler manifold has vanishing Ricci tensor, this follows from a theorem of Yau [34].

REMARKS. (1) From (9.10) the Kähler forms ω_2 and ω_3 are cohomologous to zero. The form ω_1 is not, as it restricts to a Kähler form on N_0 , the moduli space of stable bundles and from (7.7) the restriction map $H^2(\mathcal{M}, \mathbb{R}) \rightarrow H^2(N_0, \mathbb{R}) \cong \mathbb{R}$ is an isomorphism.

(2) It is a consequence of being a Stein manifold that the Betti numbers b_i vanish for $i > 6g - 6$. We have already seen this of course in Theorem (7.6), using the complex structure I .

To gain a more precise description of the complex structure J we consider its effect on the infinite-dimensional hyperkählerian manifold $\mathcal{A} \times \Omega$, from which the structure on \mathcal{M} was derived by a quotient construction. Recall that the tangent space at a point of \mathcal{A} is the space $\Omega^{0,1}(M; \text{ad } P \otimes \mathbb{C})$ and that of Ω is $\Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C})$. Thus with respect to I , the tangent space to $\mathcal{A} \times \Omega$ is the complex space

$$\Omega^{0,1}(M; \text{ad } P \otimes \mathbb{C}) \oplus \Omega^{1,0}(M; \text{ad } P \otimes \mathbb{C}).$$

The complex structure J may be defined by

$$J(A, B) = (iB^*, -iA^*)$$

and K by

$$K(A, B) = (-B^*, A^*).$$

Now define an isomorphism $\alpha: \mathcal{A} \times \Omega \rightarrow \mathcal{A} \times \bar{\mathcal{A}}$ by

$$\alpha(A, \Phi) = (d''_A + \Phi^*, d'_A + \Phi),$$

where $d_A = d'_A + d''_A$ is the covariant derivative of the unitary connection A . Then the derivative of α is

$$d\alpha(A, B) = (A + B^*, -A^* + B)$$

and hence

$$d\alpha(J(A, B)) = d\alpha(iB^*, -iA^*) = (iB^* + iA, iB - iA^*) = id\alpha(A, B).$$

Thus α identifies $\mathcal{A} \times \Omega$ with complex structure J with the space $\mathcal{A} \times \bar{\mathcal{A}}$ endowed with its natural complex structure. An element of $\mathcal{A} \times \bar{\mathcal{A}}$ is a pair (d'_1, d'_2) of operators or equivalently $d = d'_1 + d'_2$ is a (non-unitary) $\text{PSL}(2, \mathbb{C})$ connection.

Consider the self-duality equations

$$\left. \begin{aligned} d''_A \Phi &= 0, \\ F(A) + [\Phi, \Phi^*] &= 0, \end{aligned} \right\} \tag{9.11}$$

under this isomorphism. Firstly,

$$d^2 = (d'_1 + d''_2)^2 = (d'_A + \Phi)(d''_A + \Phi^*) = 0;$$

thus the $\mathrm{PSL}(2, \mathbb{C})$ connection is *flat*. Secondly, using the unique unitary connections d_1 and d_2 compatible with d''_1 and d''_2 , we have

$$\begin{aligned} F_1 = d_1^2 &= (d'_1 + d''_1)^2 = (d'_A + \Phi + d''_A - \Phi^*)^2 \\ &= F(A) - [\Phi, \Phi^*] = -2[\Phi, \Phi^*] \end{aligned}$$

and

$$\begin{aligned} F_2 = d_2^2 &= (d'_2 + d''_2)^2 = (d'_A - \Phi + d''_A + \Phi^*)^2 \\ &= F(A) - [\Phi, \Phi^*] = -2[\Phi, \Phi^*]. \end{aligned}$$

Hence

$$F_1 = F_2. \quad (9.12)$$

Thus, instead of the holomorphic equation $d''_A \Phi = 0$ and the unitary condition $F(A) + [\Phi, \Phi^*] = 0$, we have the holomorphic condition $F = 0$ for a complex connection and the unitary condition $F_1 = F_2$. As a consequence of the self-duality equations, we showed in Theorem (2.1) that the pair (V, Φ) was stable; now we have a similar stability condition for the complex connection $d = d'_1 + d''_2$. This is expressed in:

THEOREM (9.13). *Let (A, Φ) be a solution to the self-duality equations on a compact Riemann surface. Then if (A, Φ) is irreducible, so is the flat $\mathrm{PSL}(2, \mathbb{C})$ connection $d'_A + d''_A + \Phi + \Phi^*$.*

Proof. Assume the flat connection $d'_A + d''_A + \Phi + \Phi^*$ is reducible. Then its holonomy lies inside the upper triangular subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Since this preserves a 1-dimensional subspace of the Lie algebra of $\mathrm{PSL}(2, \mathbb{C})$, there is a complex line bundle L contained in $\mathrm{ad} P \otimes \mathbb{C} = W$ which is preserved by the connection, and hence inherits a flat connection d_L .

We first solve the equations $F_1 = F_2$ for a flat connection on the line bundle L equivalent under non-unitary gauge transformations to $d_L = d'_1 + d''_2$. To do this take a gauge transformation of L given by multiplication by e^u , with $u \in C^\infty(M)$. Then d_L is transformed to

$$d'_1 + d'u + d''_2 + d''u$$

and if \bar{F}_1, \bar{F}_2 are the curvatures of the unitary connections $d'_1 + d''_1, d'_2 + d''_2$, then the curvature F_1 of the unitary connection compatible with $d'_1 + d'u$, that is, $d'_1 + d''_1 + d'u - d''u$, is

$$F_1 = \bar{F}_1 - 2d'd''u. \quad (9.14)$$

The curvature F_2 of the unitary connection compatible with $d''_2 + d''u$, that is, $d'_2 + d''_2 + d''u - d'u$, is

$$F_2 = \bar{F}_2 + 2d'd''u. \quad (9.15)$$

Hence $F_1 = F_2$ is solved if we take

$$4d'd''u = \bar{F}_1 - \bar{F}_2$$

which is clearly possible since $\bar{F}_1 - \bar{F}_2$ is cohomologous to zero.

Now let $s_0 \in \Omega^0(M; L^*W)$ be the covariant constant section given by the inclusion $L \subset W$. Then $e^{-u}s_0 = s$ is covariant constant with respect to a flat complex connection on L^*W which arises as a solution of the self-duality equations on L^*W . In other words, taking the tensor product of the given connection W and the connection constructed above on L^* we have a flat connection $d = d'_1 + d''_2$ on L^*W such that $F_1 = F_2$.

Since $d''_2s = 0$, we have from (2.4),

$$\int_M \langle d'_2s, d'_2s \rangle = \int_M \langle F_2s, s \rangle. \tag{9.16}$$

But $d'_1s = 0$, and then we obtain analogously

$$\int_M \langle d''_1s, d''_1s \rangle = - \int_M \langle F_1s, s \rangle. \tag{9.17}$$

However, if $F_1 = F_2$, we have

$$0 \leq \|d'_2s\|_{L^2}^2 = - \|d''_1s\|_{L^2}^2,$$

whence $d'_2s = 0$ and $d''_1s = 0$. This means that s is covariant constant with respect to the unitary connections $d'_1 + d''_1$ and $d'_2 + d''_2$. This implies that $L \subseteq W$ is invariant by the connections $d'_A + \Phi + d''_A - \Phi^*$ and $d'_A - \Phi + d''_A + \Phi^*$. In particular, it is invariant by $d'_A + d''_A$ and by Φ and Φ^* . Thus the solution (A, Φ) of the self-duality equations is reducible.

Note that an obvious repetition of the second part of the proof gives the following proposition.

PROPOSITION (9.18). *Let (d'_1, d''_2) and $(\tilde{d}'_1, \tilde{d}''_2)$ be flat $\text{PSL}(2, \mathbb{C})$ connections arising as above from two irreducible solutions to the $\text{SO}(3)$ self-duality equations (A, Φ) and $(\tilde{A}, \tilde{\Phi})$. Then if (d'_1, d''_2) and $(\tilde{d}'_1, \tilde{d}''_2)$ are equivalent under complex gauge transformations, $(A, \tilde{\Phi})$ and (\tilde{A}, Φ) are gauge equivalent under $\text{SO}(3)$ gauge transformations.*

Proof. Put the product connection on $V \otimes V^*$ and apply the above argument to the covariant constant section which is the complex gauge transformation.

Consider now what we have achieved by looking at the moduli space \mathcal{M} from the point of view of the complex structure J . We have seen that each solution to the self-duality equations gives rise to a flat complex connection, and secondly that a vanishing theorem implies that the connection is necessarily irreducible. Compare this with what we did in § 2. There we considered the self-duality equations from the point of view of the complex structure I , showed they gave rise to a holomorphic vector bundle and a holomorphic Higgs field, and then used a vanishing theorem to prove that the pair was stable. In § 4 we showed that every stable pair arises in this way, and we may expect an analogous statement for the viewpoint considered here: every irreducible flat connection arises from a solution to the self-duality equations. This is the theorem of Donaldson [8], proved in the paper following this one:

THEOREM (9.19) (S. K. Donaldson). *Let P be a principal $SO(3)$ bundle over a compact Riemann surface M . For any irreducible flat connection on P^c there is a gauge transformation taking it to a $PSL(2, \mathbb{C})$ connection $A + \psi$ where (A, ψ) satisfy the self-duality equations.*

From Proposition (9.18) the gauge transformation is unique modulo $SO(3)$ gauge transformations.

As a consequence of Donaldson’s theorem, we may identify the complex manifold (\mathcal{M}, J) as a moduli space of flat connections or representations of the fundamental group. Recall that in order to obtain a smooth moduli space in § 5, we had to use $SU(2)$ gauge transformations modulo ± 1 rather than $SO(3)$ gauge transformations. Similarly (\mathcal{M}, J) is not the space of equivalence classes of flat $PSL(2, \mathbb{C})$ connections but is a covering instead. This can be described by using a central extension of the fundamental group $\pi_1(M)$, as did Atiyah and Bott in [5].

The group $\pi_1(M)$ is generated by $2g$ generators $A_1, B_1, \dots, A_g, B_g$ satisfying the relation

$$\prod_{i=1}^g [A_i, B_i] = 1.$$

There is a universal central extension Γ ,

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \pi_1(M) \rightarrow 1$$

generated by A_i, B_i , and central J subject to the relation $\prod_{i=1}^g [A_i, B_i] = J$.

Representations of Γ into $SL(2, \mathbb{C})$ are of two types depending on whether the central element $1 \in \mathbb{Z} \subset \Gamma$ goes to $+1$ or -1 in $SL(2, \mathbb{C})$. In the first case the representation is simply obtained from a homomorphism from $\Gamma/\mathbb{Z} = \pi_1(M)$ into $SL(2, \mathbb{C})$. The second case, the *odd* one is the situation giving rise to a flat $PSL(2, \mathbb{C})$ connection with non-zero Stiefel–Whitney class w_2 , and this is the case considered here arising from the self-duality equations on a bundle of odd degree.

Thus the complex manifold (\mathcal{M}, J) is naturally identified with

$$\text{Hom}(\Gamma, SL(2, \mathbb{C}))^{\text{odd,irr}}/SL(2, \mathbb{C}),$$

the quotient space of odd irreducible homomorphisms of Γ to $SL(2, \mathbb{C})$ modulo conjugation by $SL(2, \mathbb{C})$.

The existence of this space as a smooth $(6g - 6)$ -dimensional complex manifold is already known [13]. By its very description in terms of matrices A_i, B_i , satisfying $\prod [A_i, B_i] = -1$ it is an *affine variety*, in which case the conclusions (ii) and (iii) of Proposition (9.1) are obvious. We include them however because they are derived from very general considerations concerning hyperkähler manifolds.

Using Theorem (7.6) we deduce the next theorem.

THEOREM (9.20). *Let Γ be the universal central extension of the fundamental group of a Riemann surface of genus g by \mathbb{Z} . Then the space*

$$\text{Hom}(\Gamma, SL(2, \mathbb{C}))^{\text{odd,irr}}/SL(2, \mathbb{C})$$

is a smooth manifold which is connected, simply-connected, and has Poincaré

polynomial

$$P(t) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}[(1+t^2)^2(1+t)^{2g} - (1+t)^4(1-t)^{2g}]}{4(1-t^2)(1-t^4)} - (g-1)t^{4g-3} \frac{(1+t)^{2g-2}}{(1-t)} + 2^{2g-1}t^{4g-4}[(1+t)^{2g-2} - (1-t)^{2g-2}].$$

Note finally that whereas the complex structure I of \mathcal{M} depends on the modulus of the Riemann surface (it contains as a fixed point set of the circle action a holomorphic copy of the Riemann surface itself), the complex structure J depends only on the fundamental group and is thus independent of the conformal structure of M .

10. The real structure

We return for the moment to the moduli space \mathcal{M} with its complex structure I . We saw that $\Phi \rightarrow e^{i\theta}\Phi$ defined a circle action which preserved the metric and the complex structure. Let σ denote the action of $-1 \in U(1)$. Then σ is a holomorphic involution with respect to I . However, the action of the circle on the holomorphic symplectic form $\omega_2 + i\omega_3$ was given by $e^{i\theta}(\omega_2 + i\omega_3)$. Thus

$$\left. \begin{aligned} \sigma^* \omega_2 &= -\omega_2, \\ \sigma^* \omega_3 &= -\omega_3. \end{aligned} \right\}$$

Reverting now to the complex structure J defined by the Kähler form ω_2 we see that

$$\left. \begin{aligned} \sigma^* J &= -J, \\ \sigma^*(\omega_1 + i\omega_3) &= (\omega_1 - i\omega_3). \end{aligned} \right\} \tag{10.1}$$

Thus σ is an anti-holomorphic involution (a *real structure* on (\mathcal{M}, J)) taking the holomorphic symplectic form $\omega_1 + i\omega_3$ to its complex conjugate.

We may investigate the real points of (\mathcal{M}, J) (the fixed points of σ) by considering the action of σ on (\mathcal{M}, I) whose complex structure we know is given by the moduli space of stable pairs (V, Φ) .

Suppose then that the gauge-equivalence class of (A, Φ) is fixed by σ . If the pair (A, Φ) is fixed by σ , then $\Phi = -\Phi = 0$, so we have the moduli space of flat $SO(3)$ connections. If not, then σ acts as a gauge transformation which preserves A , and hence A must be reducible to a $U(1)$ connection: a direct sum connection on a vector bundle

$$V = L \oplus L^*(\Lambda^2 V).$$

Now an element $g \in SO(3)$ of order 2 must be a rotation by π which lifts to an element of order 4 in $SU(2)$. Thus σ acts on V via a transformation of the form

$$\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with respect to the above decomposition. The action of this by conjugation on

$\text{End}_0 V$ is

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}.$$

Hence $\sigma^*\Phi = -\Phi$ if $a = 0$.

Using this we obtain the following description of the fixed points of σ .

PROPOSITION (10.2). *Let (M, I) be the moduli space of solutions to the self-duality equations on a rank-2 vector bundle of odd degree and fixed determinant over a compact Riemann surface of genus $g > 1$. If σ is the involution induced by $(A, \Phi) \mapsto (A, -\Phi)$ then the fixed points of σ consist of complex submanifolds $\mathcal{M}_0, \mathcal{M}_{2d-1}$ ($1 \leq d \leq g - 1$) each of dimension $3g - 3$ where*

- (i) \mathcal{M}_0 is isomorphic to the moduli space of stable rank-2 bundles of fixed determinant and odd degree,
- (ii) \mathcal{M}_{2d-1} is the total space of a holomorphic vector bundle E over a 2^{2g} -fold covering of the symmetric product $S^{2g-2d-1}M$. The bundle E is the pull back to the covering of the direct image sheaf under projection of the bundle $U^{-1}K_M^2$ over $M \times S^{2g-2d-1}M$ where U is the tautological line bundle over the product, i.e. the pull-back of the Poincaré bundle on $M \times \text{Jac}^{2g-2d-1}(M)$ via the canonical map $M \times S^{2g-2d-1}M \rightarrow M \times \text{Jac}^{2g-2d-1}(M)$.

Proof. The argument follows directly the analysis of the fixed points of the circle action as in (7.1). As discussed above, if (A, Φ) is fixed by σ , then A is flat and we have as a component of the fixed point set the moduli space of flat $\text{SO}(3)$ connections which may be identified with the moduli space of stable rank-2 vector bundles of odd degree and fixed determinant. Otherwise, if the equivalence class only is fixed, then we have a stable pair (V, Φ) where

$$V \cong L \oplus L^* \Lambda^2 V$$

and $\text{deg } L = d$ say, and with respect to this decomposition

$$\Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

where $b \in H^0(M ; L^2K \otimes \Lambda^2 V^*)$ and $c \in H^0(M ; L^{-2}K \otimes \Lambda^2 V)$. Let

$$\text{deg } L - \frac{1}{2} \text{deg } \Lambda^2 V = d - \frac{1}{2} > 0;$$

then by stability $c \neq 0$. Moreover, the group of automorphisms of V is \mathbb{C}^* which acts on c by scalar multiplication. Consequently the \mathcal{G}^c orbit of (V, Φ) determines an effective divisor in the linear system $L^{-2}K \otimes \Lambda^2 V$ of degree

$$2g - 2 - 2d + 1 = 2g - 2d - 1$$

and hence a point x of $S^{2g-2d-1}M$. The fibre over this point is the choice of b and since

$$\begin{aligned} \text{deg}(L^2K \otimes \Lambda^2 V^*) &= 2g - 2 + 2d - 1 \\ &= 2g + 2d - 3 \\ &> 2g - 2, \end{aligned}$$

then

$$\dim H^0(M ; L^2K \otimes \Lambda^2 V^*) = g - 1 + 2d - 1$$

making the whole set of equivalence classes a vector bundle of rank $g + 2d - 2$ over $S^{2g-2d-1}M$ and hence a complex manifold of dimension $3g - 3$.

The line bundle $U_x = L^{-2}K \otimes \Lambda^2V$ is the bundle of the divisor corresponding to a point x of $S^{2g-2d-1}M$ and the fibre over x is

$$H^0(M ; L^2K \otimes \Lambda^2V^*) = H^0(M ; U_x^{-1}K^2),$$

hence the description of E in the proposition.

REMARK. Topological properties of the bundle E over $S^{2g-2d-1}M$ may be deduced by applying the Grothendieck Riemann–Roch theorem to

$$M \times \text{Jac}^{2g-2d-1}(M)$$

with respect to the projection

$$p: M \times \text{Jac}(M) \rightarrow \text{Jac}(M).$$

This gives

$$p_*(ch(U^{-1}K^2) \cdot td(M \times J(M))) = ch(p_!(U^{-1}K^2)) \cdot td(J(M)). \tag{10.3}$$

Now

$$H^1(M ; U^{-1}K^2) = H^1(M ; L^2K \otimes \Lambda^2V^*) = 0$$

since $\text{deg}(L^2K \otimes \Lambda^2V^*) > 2g - 2$, so

$$ch(p_!(U^{-1}K^2)) = ch(\tilde{E}) \quad \text{where } \tilde{E} = p_*U^{-1}K^2$$

and therefore, since the tangent bundle of the Jacobian is trivial,

$$ch(\tilde{E}) = p_*(ch(U^{-1}K^2)td(M)). \tag{10.4}$$

Now (see [1]), $c_1(U) = (2g - 2d - 1)\eta + \gamma$ where η is the pull-back of the class of a point on M and $\gamma \in H^1(M, \mathbb{Z}) \otimes H^1(\text{Jac}(M), \mathbb{Z})$ the canonical element. Hence

$$\begin{aligned} & p_*(ch(U^{-1}K^2)td(M)) \\ &= p_*((1 - (2g - 2d - 1)\eta - \gamma + \frac{1}{2}((2g - 2d - 1)\eta + \gamma)^2) \\ & \quad \times (1 + (4g - 4)\eta)(1 - (g - 1)\eta)) \\ &= (g + 2d - 2) - \theta, \end{aligned}$$

where $\theta \in H^2(\text{Jac}(M), \mathbb{Z})$ is the class of the theta-divisor.

Hence

$$ch(\tilde{E}) = (g + 2d - 2) - \theta. \tag{10.5}$$

The Chern classes are then

$$c_k(\tilde{E}) = (-1)^k \theta^k / k!.$$

These are in particular, as θ defines a principal polarization, *primitive* classes in $H^{2k}(\text{Jac}(M), \mathbb{Z})$. Now from [23], the cohomology homomorphism

$$H^r(S^n M, \mathbb{Z}) \rightarrow H^r(S^{n-1} M, \mathbb{Z}),$$

induced by the inclusion $S^{n-1}M \subset S^n M$ given by adding a fixed point, is an isomorphism for $0 \leq r \leq n - 1$ and injective for $r = n - 1$. Now $S^n M$ is a projective bundle over $\text{Jac}(M)$ for $n > 2g = 2$ and hence by the Leray–Hirsch theorem $H^*(S^n M, \mathbb{Z})$ is a free module over $H^*(\text{Jac}(M), \mathbb{Z})$ generated by an element

$X \in H^2(S^n M, \mathbb{Z})$. In particular, a primitive class in $H^{2k}(\text{Jac}(M), \mathbb{Z})$ is primitive in $H^{2k}(S^n M, \mathbb{Z})$.

Hence from the map $f: S^{2g-2d-1}M \rightarrow S^n M \rightarrow \text{Jac}(M)$ we see that $E = f^* \bar{E}$ has Chern classes $c_k(E)$ which are primitive if $2k < 2g - 2d - 1$. In particular, $c_1(E)$ is primitive if $d < g - 1$, and so $w_2(E) = c_1(E) \bmod 2$ is non-zero. Thus the bundle E , even as a real vector bundle, is non-trivial if $d < g - 1$.

If $d = g - 1$ we have a bundle over the Riemann surface M itself whose first Chern class is

$$c_1(E) = -\theta.$$

To evaluate on M , recall that $M \subset \text{Jac}(M)$ is the space of special divisors W_1 which has cohomology class $\theta^{g-1}/(g-1)!$. Hence

$$\text{deg } E = -\theta^g[\text{Jac}(M)]/(g-1)! = -g. \tag{10.6}$$

As a real rank $6g - 8$ bundle on M , E is therefore trivial if and only if $w_2 = 0$, which from (10.6) occurs only if g is even.

Consider now the map

$$\alpha: \mathcal{M} \rightarrow \mathcal{H}$$

taking (A, Φ) to the equivalence class of the flat $\text{PSL}(2, \mathbb{C})$ connection

$$d'_A + d''_A + \Phi + \Phi^*.$$

Since σ is an anti-holomorphic involution on (\mathcal{M}, J) and α is holomorphic with respect to this real structure, the connection $\alpha(A, \Phi)$ for a pair (A, Φ) which represent a fixed point of σ must possess a reality constraint.

If $(A, \Phi) \in \mathcal{M}_0$, that is, $\Phi = 0$, then this is clearly true, for $d'_A + d''_A$ is then a flat $\text{SO}(3)$ connection. Suppose, however that $(A, \Phi) \in \mathcal{M}_{2d-1}$ for $d \geq 1$; then A defines a $\text{U}(1)$ connection on

$$V = L \oplus L^* \Lambda^2 V.$$

We define an antilinear homomorphism

$$T: V \rightarrow V \otimes \Lambda^2 V^*$$

by

$$T(u_1, u_2) = (\bar{u}_2, \bar{u}_1) \tag{10.7}$$

where we use the unitary structure on L to identify

$$\bar{L} \cong L^*.$$

Consider $\Phi \in H^0(M; \text{End}_0 V \otimes K)$ of the form

$$\Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \Phi T(u_1, u_2) &= (b\bar{u}_1, c\bar{u}_2) \\ &= T(\bar{c}u_2, \bar{b}u_1) \\ &= T\Phi^*(u_1, u_2). \end{aligned}$$

Thus the connection $d'_A + d''_A + \Phi + \Phi^*$ commutes with T . On the projective bundle $P(V)$, T induces a real structure with real points (u, \bar{u}) in homogeneous

coordinates. These real points give an $\mathbb{R}P^1$ -bundle over M associated to the flat $\mathrm{PSL}(2, \mathbb{R})$ connection $d'_A + d''_A + \Phi + \Phi^*$. Thus each of the other components \mathcal{M}_{2d-1} ($d > 0$) of the fixed point set of σ correspond under α to the space of equivalence classes of flat $\mathrm{PSL}(2, \mathbb{R})$ connections

$$\mathrm{Hom}(\pi_1, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R}).$$

There remains the interpretation of the degree $0 < d \leq g - 1$. Note however that the map

$$(u, \bar{u}) \rightarrow u/\bar{u}$$

defines an isomorphism from the $\mathbb{R}P^1$ bundle associated to the $\mathrm{PSL}(2, \mathbb{R})$ connection to the unit vectors in the complex line bundle $L^2 \otimes \Lambda^2 V^*$. Thus the Euler class of the $\mathbb{R}P^1$ bundle is equal to the degree

$$\mathrm{deg}(L^2 \otimes \Lambda^2 V^*) = 2d - 1.$$

The restriction to odd Euler class here is not fundamental. We have used the principal bundle P with $w_2 \neq 0$ throughout only because the moduli space of solutions on the corresponding vector bundle is smooth. If $w_2 = 0$, then there may exist solutions to the self-duality equations reducible to $U(1)$. These give rise, however, to vector bundles $V = L \oplus L^*$ where L is flat and hence of degree 0. Thus the solutions corresponding as above to flat $\mathrm{PSL}(2, \mathbb{R})$ connections with non-zero Euler class correspond to smooth points in the moduli space. Equally, even in the case where $w_2 \neq 0$, we were forced to use $SU(2)$ gauge transformations instead of $SO(3)$ transformations because of fixed points of the action of tensoring V by a line bundle L of order 2. If the connection on V is reducible, so that $V = L_1 \oplus L_2$, then $V \cong V \otimes L$ implies $L_2 \cong L_1 \otimes L$ if L_1 and L_2 themselves are not isomorphic, and hence in particular $\mathrm{deg} L_1 = \mathrm{deg} L_2$. This again is the situation of zero Euler class.

It follows that we can define moduli spaces \mathcal{M}_k for $0 < k \leq 2g - 2$ which are fixed points of the involution induced by $\sigma: (A, \Phi) \rightarrow (A, -\Phi)$ and on which \mathbb{Z}_2^g acts freely. The method of Proposition (10.2) shows that $\mathcal{M}_k/\mathbb{Z}_2^g$ is the total space of a rank $(g - 1 + k)$ complex vector bundle over the symmetric product $S^{2g-2-k}M$. On the other hand, by Donaldson's theorem (9.19) this space is diffeomorphic to the moduli space of flat $\mathrm{PSL}(2, \mathbb{R})$ connections of Euler class $k > 0$ (these are all irreducible as reducibility would give a section of the associated $\mathbb{R}P^1$ bundle, thus forcing $k = 0$). We therefore have the following theorem.

THEOREM (10.8). *Let $\pi_1(M)$ be the fundamental group of a compact Riemann surface of genus $g > 1$, and let $\mathrm{Hom}(\pi_1, \mathrm{PSL}(2, \mathbb{R}))^k$ denote the space of homomorphisms of π_1 to $\mathrm{PSL}(2, \mathbb{R})$ whose associated $\mathbb{R}P^1$ bundle has Euler class k . Then the quotient space $\mathrm{Hom}(\pi_1, \mathrm{PSL}(2, \mathbb{R}))^k/\mathrm{PSL}(2, \mathbb{R})$ is a smooth manifold of dimension $(6g - 6)$ which is diffeomorphic to a complex vector bundle of rank $(g - 1 + k)$ over the symmetric product $S^{2g-2-k}M$.*

COROLLARY (10.9) (Milnor and Wood [24, 33]). *The Euler class k of any flat $\mathrm{PSL}(2, \mathbb{R})$ bundle satisfies $|k| \leq 2g - 2$.*

Proof. By Donaldson's theorem (9.19) the flat connection arises from a

solution of the self-duality equations invariant under σ , and hence in particular giving a holomorphic vector bundle $V = L_1 \oplus L_2$. By Remark (3.12), the stability condition for the pair (V, Φ) gives $\text{deg } L_1 - \text{deg } L_2 \leq (2g - 2)$.

COROLLARY (10.10) (Goldman [12]). *If $g = 2$ and $k = 1$, then*

$$\text{Hom}(\pi_1, \text{PSL}(2, \mathbb{R}))^1 / \text{PSL}(2, \mathbb{R}) \cong M \times \mathbb{R}^4.$$

Proof. From (10.6), $w_2(E) = 0$, so as a real vector bundle, E is trivial.

In the case where $k = 2g - 2$, this moduli space is simply a vector space of complex dimension $(3g - 3)$. We consider this component and its special interpretation in the next section.

11. Teichmüller space

The main advantage throughout this paper in staying with the group $\text{SO}(3)$ was to simplify the algebra and analysis. However, it also provided a link with the underlying geometry of M , for we saw in Example (1.5) that the metric of constant curvature -4 , compatible with the underlying conformal structure, had a description in terms of a solution to the self-duality equations. We shall see now how *all* metrics of constant negative curvature can equally be described even though we fix the complex structure of M .

We consider the moduli space \mathcal{M}_{2g-2} of solutions to the $\text{SO}(3)$ self-duality equations on a principal bundle P with $w_2(P) = 0$, which are fixed points of the involution σ and correspond to a line bundle of degree $(g - 1)$. In other words, we have a $\text{U}(1)$ connection on

$$V = L \oplus L^*$$

where $\text{deg } L = g - 1$ and a Higgs field

$$\Phi = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in H^0(M; \text{End}_0 V \otimes K).$$

From the proof of (10.2), or stability of (V, Φ) in general, we must have

$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$$

and $b = 1 \in H^0(M; \text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}})K)$, after normalizing with the \mathbb{C}^* group of automorphisms of V . With this normalization,

$$a \in H^0(M; \text{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}})K) = H^0(M; K^2)$$

is a *quadratic differential* and $\mathcal{M}_{2g-2} \cong H^0(M; K^2) \cong \mathbb{C}^{3g-3}$.

The self-duality equations

$$F + [\Phi, \Phi^*] = 0$$

now become the abelian *vortex equations* (see [19]),

$$F_1 = 2(1 - \|a\|^2)\omega \tag{11.1}$$

where F_1 is the curvature of the $\text{U}(1)$ connection on K and $\omega \in \Omega^{1,1}(M)$ is the Kähler form of the corresponding metric on M . Note that the case where $a = 0$ is the metric of constant negative curvature discussed in (1.5).

THEOREM (11.2). *Let M be a compact Riemann surface of genus $g > 1$ and $a \in H^0(M; K^2)$ a quadratic differential. Let (A, Φ) be the solution to the $SU(2)$ self-duality equations on M from Theorem (4.3) for which*

$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

Then

- (i) *if h is the hermitian form on K^{-1} determined by the unitary structure preserved by A , then*

$$\hat{h} = a + \left(h + \frac{a\bar{a}}{h} \right) + \bar{a} \in \Omega^0(M; S^2T^* \otimes \mathbb{C})$$

is a Riemannian metric on M ,

- (ii) *the Gaussian curvature of \hat{h} is -4 ,*
- (iii) *any metric of constant curvature -4 on M is isometric to a metric of this form for some $a \in H^0(M; K^2)$.*

Proof. (i) The bundle $S^2T^* \otimes \mathbb{C}$ is

$$S^2(K \oplus \bar{K}) \cong K^2 \oplus K\bar{K} \oplus \bar{K}^2$$

with real structure

$$(a, b, c) \rightarrow (\bar{c}, \bar{b}, \bar{a}).$$

Clearly $\hat{h} = a + (h + (a\bar{a}/h)) + \bar{a}$ is real and thus describes a real symmetric form on TM . We must prove that it is non-degenerate, i.e. that

$$\frac{1}{4} \left(h + \frac{a\bar{a}}{h} \right)^2 - a\bar{a} > 0,$$

that is, $(h - (a\bar{a}/h))^2 > 0$ for all $x \in M$.

In terms of the vortex equations (11.1) we need to show that the norm of the Higgs field a is everywhere less than 1. This is an important property of both vortices and monopoles in Euclidean space and we prove it using the same method as in [19]—the strong maximum principle.

The quadratic differential satisfies $d''a = 0$ and so, following the development of (2.1), we have

$$d'' \langle d'a, a \rangle = \langle Fa, a \rangle - \langle d'a, d'a \rangle$$

or using (11.1),

$$d'' d' \|a\|^2 = 4(1 - \|a\|^2) \|a\|^2 \omega - \|d'a\|^2 \omega,$$

that is,

$$\Delta \|a\|^2 = 4(1 - \|a\|^2) \|a\|^2 - \|d'a\|^2, \tag{11.3}$$

where the Laplacian Δ is a positive operator.

Using the strong maximum principle [19, VI.3] to $\mathcal{L} = -\Delta - 4 \|a\|^2$ applied to $1 - \|a\|^2$ we see that $\|a\| < 1$ for all x , that is,

$$a\bar{a}/h^2 < 1,$$

thus showing that \hat{h} is a metric.

(ii) We may write $a = q dz^2$ in local coordinates and then

$$\hat{h} = h \left(dz + \frac{\bar{q} d\bar{z}}{h} \right) \left(d\bar{z} + \frac{q dz}{h} \right).$$

Hence $u = dz + (\bar{q}/h) d\bar{z}$ is a $(1, 0)$ form with respect to \hat{h} such that

$$\hat{h}(u, u) = h^{-1}. \quad (11.4)$$

Let $\hat{\nabla}$ be the Levi-Civita connection of \hat{h} ; then $\hat{\nabla}u = u \otimes \theta$, for a 1-form θ . Since the connection is torsion-free, $du = \theta \wedge u$. But $du = d(dz + (\bar{q}/h) d\bar{z})$. Therefore, since q is holomorphic,

$$\theta \wedge u = -\frac{\bar{q}}{h^2} \frac{\partial h}{\partial z} dz \wedge d\bar{z}. \quad (11.5)$$

The connection preserves \hat{h} , and hence

$$d(\hat{h}(u, u)) = d(h^{-1}) = (\theta + \bar{\theta})h^{-1}.$$

Hence

$$-\frac{dh}{h} = \theta + \bar{\theta}. \quad (11.6)$$

From (11.5),

$$-\theta^{0,1} + \frac{\bar{q}}{h} \theta^{1,0} = -\frac{\bar{q}}{h^2} \frac{\partial h}{\partial z},$$

and from (11.6),

$$\bar{\theta}^{0,1} + \theta^{1,0} = -\frac{1}{h} \frac{\partial h}{\partial z}.$$

Hence, if $\theta^{0,1} \neq 0$,

$$\frac{\bar{q}}{h} \theta^{0,1} = -\bar{\theta}^{0,1},$$

so that $q\bar{q} = h^2$. This however is impossible since $a = q dz^2$ has zeros; thus $\theta^{0,1} = 0$ and

$$\theta^{1,0} = -\frac{1}{h} \frac{\partial h}{\partial z}. \quad (11.7)$$

Thus the curvature of $\hat{\nabla}$ on the bundle of $(1, 0)$ forms is

$$\hat{F} = d\theta^{1,0} = -d''d' \log h. \quad (11.8)$$

Now $h dz d\bar{z}$ is the original metric on M , so

$$F = d'd'' \log h = 2 \left(h - \frac{a\bar{a}}{h} \right) dz d\bar{z}. \quad (11.9)$$

The Kähler form of the metric \hat{h} is

$$\hat{\omega} = h \left(dz + \frac{\bar{q}}{h} d\bar{z} \right) \wedge \left(d\bar{z} + \frac{q}{h} dz \right) = \left(h - \frac{q\bar{q}}{h} \right) dz d\bar{z}.$$

Thus from (11.8) and (11.9),

$$\hat{F} = 2\hat{\omega},$$

so \hat{h} is a metric of constant curvature -4 .

(iii) Suppose now that g_0 is the constant-curvature metric compatible with the complex structure of M , and g another metric with the same constant curvature. We use the Earle and Eells approach to Teichmüller theory [9] next. Since (M, g) has negative curvature, it follows from the Eells–Sampson existence theorem that there is a unique harmonic *diffeomorphism*

$$\phi: (M, g_0) \rightarrow (M, g)$$

which is homotopic to the identity and minimizes the energy

$$E = \int_M \|d\phi\|^2$$

among all such maps.

Since ϕ is harmonic, the $(2, 0)$ part of the metric ϕ^*g is holomorphic with respect to g_0 , that is,

$$\phi^*g = a + b + \bar{a}$$

where $a \in H^0(M, K^2)$ is a quadratic differential and $b \in \Omega^{1,1}(M)$.

Since ϕ^*g is a metric, $b^2 - 4a\bar{a} > 0$ for all $x \in M$, and hence the root

$$h = \frac{1}{2}(b + \sqrt{b^2 - 4a\bar{a}})$$

to the equation $h + (a\bar{a}/h) = b$ is everywhere positive and thus defines a metric on M compatible with its complex structure.

Reversing the calculations of (ii) and using the fact that ϕ^*g has constant curvature -4 , we recover the self-duality equations (11.1).

Consequently every constant-curvature metric on M is isometric to a metric constructed from a solution to the self-duality equations.

Recall that Teichmüller space \mathcal{T} is the space of equivalence classes of metrics of constant curvature -4 on M , modulo the action of the group of diffeomorphisms of M homotopic to the identity. Then, as a consequence of (11.2) we have the following corollary.

COROLLARY (11.10). *Teichmüller space is homeomorphic to \mathbb{R}^{6g-6} .*

Proof. Take the homeomorphism from $H^0(M; K^2) = \mathbb{R}^{6g-6}$ to \mathcal{T} given in Theorem (11.2).

REMARKS. (1) The energy of the harmonic map ϕ in the theorem is

$$E(\phi) = \int_M b.$$

Put in terms of the metric h , this is given by

$$E(\phi) = \int_M (1 + \|a\|^2)\omega_h$$

since $b = h + (a\bar{a}/h)$. However, as a stable pair, we have

$$\Phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

and so $\|\Phi\|_{L^2}^2 = \int_M (1 + \|a\|^2)\omega_h = E(\phi)$.

Thus the function $\mu = \|\Phi\|_{L^2}^2$ which we used in § 7 as a Morse function to compute the Betti numbers of the moduli space restricts to be simply the energy of the harmonic map.

It is interesting to note that a proof of Tromba that Teichmüller space is a ball (see [20]) uses $E(\phi)$ as a Morse function with just one critical point.

(2) Our version of Teichmüller space \mathcal{T} is \mathbb{C}^{3g-3} as a complex manifold and has a complete Kähler metric with a circle action. None of these properties holds for the usual complex structure on \mathcal{T} . This is perhaps not surprising as our description depends on a base point determined by the given complex structure on M . Symplectically, however, the two models coincide as we shall see later.

Consider finally the flat $SL(2, \mathbb{R})$ connection determined by a solution to the self-duality equations under consideration:

$$\left. \begin{aligned} d''a &= 0, \\ F_1 &= 2(1 - \|a\|^2). \end{aligned} \right\}$$

If M has a metric of constant negative curvature -4 , then the universal covering \tilde{M} is isometric to the upper half-space, and we obtain from the covering transformations a homomorphism

$$\gamma: \pi_1(M) \rightarrow PSL(2, \mathbb{R}) \tag{11.11}$$

which in fact lifts to $SL(2, \mathbb{R})$. By its very construction, the Euler class of the corresponding flat bundle is $2g - 2$.

We want to show that the flat $PSL(2, \mathbb{R})$ connection $\alpha(A, \Phi)$ which corresponds to a point in $\mathcal{M}_{2g-2} \cong \mathbb{C}^{3g-3}$ is the flat connection defined above for the metric of constant curvature \hat{h} in Theorem (11.2).

Firstly, consider the upper half-space H with the metric

$$h = \frac{dz d\bar{z}}{4y^2}$$

of constant curvature -4 . The linear representation of $SL(2, \mathbb{R})$ is defined in terms of the differential geometry of H by considering the half order differentials:

$$\phi = (az + b) dz^{-\frac{1}{2}}.$$

The natural action of $SL(2, \mathbb{R})$ on this 2-dimensional space of differentials is the standard 2-dimensional representation. It is real with respect to the ordinary conjugation operation. Since ϕ is linear in Z , another way of describing ϕ is as a section of the bundle of 1-jets

$$0 \rightarrow K^{-\frac{1}{2}} \otimes K \rightarrow J_1(K^{-\frac{1}{2}}) \rightarrow K^{-\frac{1}{2}} \rightarrow 0 \tag{11.12}$$

of sections of $K^{-\frac{1}{2}}$ which is covariant constant with respect to a certain flat connection. Using the connection of the constant-curvature metric, (11.12) can be split as a direct sum

$$J_1(K^{-\frac{1}{2}}) \cong K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$$

and the flat connection written in terms of the Levi-Civita connection.

Thus if $\phi = (az + b) dz^{-\frac{1}{2}}$, then using the Levi-Civita connection we have

$\nabla^{1,0} dz = dz^2/iy$ and so

$$\begin{aligned} \nabla^{1,0}\phi &= a dz^{\frac{1}{2}} + (az + b)\nabla^{1,0} dz^{-\frac{1}{2}} \\ &= a dz^{\frac{1}{2}} - (az + b) \frac{dz^{\frac{1}{2}}}{2iy}. \end{aligned}$$

Therefore, setting $(s_1, s_2) = (\nabla^{1,0}\phi, \phi)$, a section of $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, we have

$$\nabla^{1,0}(0, s_2) = (0, s_1)$$

and

$$\begin{aligned} \nabla^{1,0}s_1 &= \left(a - \frac{(az + b)}{2iy} \right) \cdot \frac{dz^{\frac{3}{2}}}{2iy} + \left(-\frac{a}{2iy} + \frac{(az + b)}{2iy^2} \frac{1}{2i} \right) dz^{\frac{3}{2}} \\ &= 0. \end{aligned}$$

Thus

$$\nabla^{1,0}(s_1, s_2) = (0, s_1),$$

and so $s = (s_1, s_2)$ satisfies

$$\left(d'_A + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) s = 0, \tag{11.13}$$

where A is the Levi-Civita connection on $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$. Using the real structure defined by the metric as in (10.7) we see that the flat $SL(2, \mathbb{R})$ connection which describes the hyperbolic structure on M is just the connection

$$d'_A + d''_A + \Phi + \Phi^*$$

for

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This shows in particular that $\alpha(A, \Phi)$ is the correct connection for $a = 0 \in H^0(M; K^2)$, that is, for the constant-curvature metric compatible with the conformal structure.

We must now extend this to non-zero a , so assume we are dealing with the constant-curvature metric

$$\hat{h} = h \left(dz + \frac{\bar{q}}{h} d\bar{z} \right) \left(d\bar{z} + \frac{q}{h} dz \right)$$

of Theorem (11.2). We saw in the proof that the $(1, 0)$ form $u = dz + (\bar{q}/h) d\bar{z}$ satisfied

$$\hat{\nabla}^{1,0}u = -\frac{1}{h} \frac{\partial h}{\partial z} u dz.$$

Consider the local section

$$s = (u^{\frac{1}{2}}, u^{-\frac{1}{2}}) \in \Omega^0(M, \hat{K}^{\frac{1}{2}} \oplus \hat{K}^{-\frac{1}{2}}).$$

Then

$$\hat{\nabla}^{1,0}s = \left(-\frac{1}{2h} \frac{\partial h}{\partial z} u^{\frac{1}{2}}, \frac{1}{2h} \frac{\partial h}{\partial z} u^{-\frac{1}{2}} \right) dz.$$

The canonical map

$$1 \in \text{Hom}(\hat{K}^{\frac{1}{2}}, \hat{K}^{-\frac{1}{2}} \otimes \hat{K})$$

is with respect to this basis given by

$$u = dz + \frac{\bar{q}}{h} d\bar{z}.$$

From (11.4), $\hat{h}(u, u) = h^{-1}$, so

$$(v, v^{-1}) = (h^{\frac{1}{2}}u^{\frac{1}{2}}, h^{-\frac{1}{2}}u^{-\frac{1}{2}})$$

is a unitary basis, and 1 is represented by

$$h^{\frac{1}{2}}u = h^{\frac{1}{2}} dz + \bar{q}h^{-\frac{1}{2}} d\bar{z}. \tag{11.15}$$

Hence the flat connection, relative to this basis, has connection form

$$\begin{pmatrix} \frac{1}{4h} \left(\frac{\partial h}{\partial \bar{z}} d\bar{z} - \frac{\partial h}{\partial z} dz \right) & h^{\frac{1}{2}} d\bar{z} + h^{-\frac{1}{2}} q dz \\ h^{\frac{1}{2}} dz + h^{-\frac{1}{2}} \bar{q} d\bar{z} & -\frac{1}{4h} \left(\frac{\partial h}{\partial \bar{z}} d\bar{z} - \frac{\partial h}{\partial z} dz \right) \end{pmatrix} \tag{11.16}$$

Now note that, relative to the metric $h dz d\bar{z}$, the unitary basis $(h^{\frac{1}{2}} dz^{\frac{1}{2}}, h^{-\frac{1}{2}} dz^{-\frac{1}{2}})$ of $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ has connection form

$$\begin{pmatrix} \frac{1}{4h} \left(\frac{\partial h}{\partial \bar{z}} d\bar{z} - \frac{\partial h}{\partial z} dz \right) & 0 \\ 0 & -\frac{1}{4h} \left(\frac{\partial h}{\partial \bar{z}} d\bar{z} - \frac{\partial h}{\partial z} dz \right) \end{pmatrix}$$

and $1 \in \text{Hom}(K^{\frac{1}{2}}, K^{-\frac{1}{2}} \otimes K)$ is represented by $h^{\frac{1}{2}} dz$. Moreover,

$$a = q dz^2 \in \text{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K)$$

is represented by $h^{-\frac{1}{2}} q dz$. The connection with form (11.16) is thus gauge equivalent to the connection

$$\nabla + \Phi + \Phi^* = d'_A + d''_A + \Phi + \Phi^*$$

via the transformation $q: K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \rightarrow \hat{K}^{\frac{1}{2}} \oplus \hat{K}^{-\frac{1}{2}}$, induced by $f: K \rightarrow \hat{K}$, $f(\alpha) = \alpha + \bar{a}\alpha/h$, where

$$\Phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \quad V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}},$$

and A is the Levi-Civita connection of $h dz d\bar{z}$.

Hence in all cases, the flat $SL(2, \mathbb{R})$ connection $d'_A + d''_A + \Phi + \Phi^*$ is equivalent to the canonical connection associated to the metric of constant curvature

$$\hat{h} = q dz^2 + \left(h + \frac{q\bar{q}}{h} \right) dz d\bar{z} + \bar{q} d\bar{z}^2.$$

REMARK. In [11], Goldman showed that every flat $PSL(2, \mathbb{R})$ connection with Euler class $2g - 2$ is the connection associated to a constant-curvature metric. This is also a corollary of Donaldson's theorem (9.19), for every such connection arises from a solution of the self-duality equations and we have just seen how these give all metrics of constant negative curvature.

One final aspect is the symplectic structure of Teichmüller space, as determined by the Weil–Petersson metric. It was shown by Goldman [11] that this is the natural symplectic structure on the moduli space of the associated flat $\text{PSL}(2, \mathbb{R})$ connections. That is, we identify the tangent space to the moduli space of flat connections with the first cohomology of the elliptic complex

$$\Omega^0(M, \text{ad } P) \xrightarrow{d_A} \Omega^1(M, \text{ad } P) \xrightarrow{d_A} \Omega^2(M, \text{ad } P)$$

for a $\text{PSL}(2, \mathbb{R})$ connection A on the principal bundle P . The skew form induced by $\int_M \text{Tr}(\dot{A} \wedge \dot{B})$ for d_A -closed representatives $\dot{A}, \dot{B} \in \Omega^1(M; \text{ad } P)$ defines the symplectic structure.

The corresponding complex symplectic structure on $\text{PSL}(2, \mathbb{C})$ connections induces via the map α the symplectic structure

$$\int_M \text{Tr}(\dot{A} + \dot{\Phi} + \dot{\Phi}^*)(\dot{B} + \dot{\Psi} + \dot{\Psi}^*).$$

However, if the connections are reducible and $\dot{\Phi}$ and $\dot{\Psi}$ of the form

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

as occurs in \mathcal{M}_{2g-2} , then this expression simplifies to

$$\int_M \text{Tr}(\dot{A}\dot{B}) + \text{Tr}(\dot{\Phi}^*\dot{\Psi}) + \text{Tr}(\dot{\Phi}\dot{\Psi}^*),$$

which is the Kähler form ω_1 restricted to the fixed point set of σ , \mathcal{M}_{2g-2} . Hence we deduce our final proposition.

PROPOSITION (11.17). *Let \mathcal{M}_{2g-2} be the moduli space of solutions of the $\text{SO}(3)$ self-duality equations on a compact Riemann surface M of genus $g > 1$, for which the corresponding stable pair (V, Φ) is of the form*

$$V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \quad \Phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix},$$

where $a \in H^0(M; K^2) \cong \mathbb{C}^{3g-3}$.

Let ω_1 be the Kähler form of the natural Kähler metric on \mathcal{M}_{2g-2} . Then $(\mathcal{M}_{2g-2}, \omega_1)$ is symplectically diffeomorphic to Teichmüller space with the Weil–Petersson symplectic form.

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